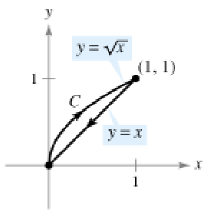
MATH152 CALCULUS II   TUTORIAL – 12

(27.05.2016)

Question 1:

Find a piecewise smooth parametrization of the path C.



1. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \le t \le 2 \end{cases}$

Question 2 :

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If $\mathbf{F}$ is conservative, the integration may be easier on an alternative path.)

$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 2$

1. Since $\mathbf{curl\ F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative.
2. The potential function is $f(x, y, z) = xyz + k$.
3. (a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$
4. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,2,0)}^{(4,2,4)}$
5. $= 32$

Question 3 :

Evaluate the line integral using the Fundamental

Theorem of Line Integrals.

$$\int_C (y + 2z)\,dx + (x - 3z)\,dy + (2x - 3y)\,dz$$

(a) C: line segment from $(0, 0, 0)$ to $(1, 1, 1)$

(b) C: line segments from $(0, 0, 0)$ to $(0, 0, 1)$ to $(1, 1, 1)$

(c) C: line segments from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$

1. $\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.
2. (a) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,1,1)} = 0 - 0$
3. $= 0$
4. (b) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(0,0,1)} + \Big[xy - 3yz + 2xz\Big]_{(0,0,1)}^{(1,1,1)} = 0 + 0$
5. $= 0$
6. (c) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,0,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,0,0)}^{(1,1,0)} +$

$\Big[xy - 3yz + 2xz\Big]_{(1,1,0)}^{(1,1,1)} = 0 + 1 + (-1)$

7. $= 0$

Question 4:

Use Green's Theorem to evaluate the integral

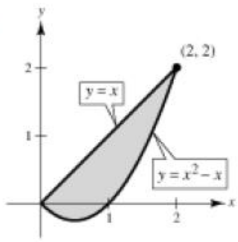
$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$

1.



2. $\displaystyle \int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2 \int_{x^2 - x}^{x} dy\,dx$

3. $\displaystyle = \int_0^2 (2x - x^2)\,dx$

4. $\displaystyle = \frac{4}{3}$

Question 5:

Use Green's Theorem to evaluate the line integral.

$$\int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy$$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

1. By Green's Theorem,

$\displaystyle \int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy = \int_R\int [(y - \sin x \sin y) - (-\sin x$

2. $\displaystyle = \int_0^1 \int_x^{\sqrt{x}} y\,dy\,dx$

3. $\displaystyle = \frac{1}{2}\int_0^1 (x - x^2)\,dx$

4. $\displaystyle = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

5. $\displaystyle = \frac{1}{12}.$

Question 6:

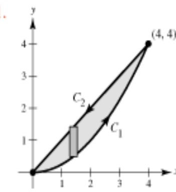
Verify Green's Theorem by evaluating both integrals

$$\int_C y^2\,dx + x^2\,dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

for the given path.

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2/4$

1.



2. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8 - t)\mathbf{i} + (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

3. $\displaystyle \int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 [(8 - t)^2(-dt) + (8 - t)^2(-dt)]$

4. $\displaystyle = \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8 - t)^2\,dt$

5. $\displaystyle = \frac{224}{5} - \frac{128}{3}$

6. $\displaystyle = \frac{32}{15}$

7. By Green's Theorem,

$\displaystyle \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4 \int_{x^2/4}^{x} (2x - 2y)\,dy\,dx$

8. $\displaystyle = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx$

9. $\displaystyle = \frac{32}{15}.$

Question : 7

Use Green's Theorem to evaluate the integral

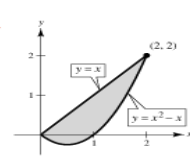
$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$

1.



2. $\displaystyle \int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2 \int_{x^2 - x}^{x} dy\,dx$

3. $\displaystyle = \int_0^2 (2x - x^2)\,dx$

4. $\displaystyle = \frac{4}{3}$

Question : 8

Evaluate $\displaystyle\iint_S (x - 2y + z)\, dS.$

S: $z = 4 - x,\ 0 \le x \le 4,\ 0 \le y \le 4$

1. $\dfrac{\partial z}{\partial x} = -1$

2. $\dfrac{\partial z}{\partial y} = 0$

3. $\displaystyle\iint_S (x - 2y + z)\, dS = \int_0^4\int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\, dy\, dx$

4. $\displaystyle= \sqrt{2}\int_0^4\int_0^4 (4 - 2y)\, dy\, dx$

5. $= 0$

Question : 9

Evaluate $\displaystyle\iint_S xy\, dS.$

S: $z = 6 - x - 2y$, first octant

1. $\dfrac{\partial z}{\partial x} = -1$

2. $\dfrac{\partial z}{\partial y} = -2$

3. $\displaystyle\iint_S xy\, dS = \int_0^6\int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx$

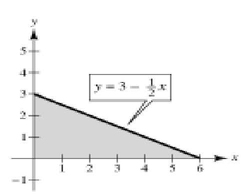
4. $\displaystyle= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$

5. $\displaystyle= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$

6. $\displaystyle= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6$

7. $\displaystyle= \frac{27\sqrt{6}}{2}$

8.



Question : 10

Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{N}\, dS$$

and find the outward flux of $\mathbf{F}$ through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

S: $x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a$

1. Since div $\mathbf{F} = 2x + 2y + 2z$, we have

2. $\displaystyle\iiint_Q \text{div}\,\mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a (2x + 2y + 2z)\, dz\, dy\, dx$

3. $\displaystyle= \int_0^a\int_0^a (2ax + 2ay + a^2)\, dy\, dx$

4. $\displaystyle= \int_0^a (2a^2x + 2a^3)\, dx$

5. $\displaystyle= \Big[a^2x^2 + 2a^3x\Big]_0^a$

6. $= 3a^4.$

Question : 11

Verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

$\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$

$S\colon\ z = \sqrt{1 - x^2 - y^2}$

1. In this case, $M = -y + z,$
2. $N = x - z,$
3. $P = x - y$ and
4. C is the circle $x^2 + y^2 = 1,$
5. $z = 0,$
6. $dz = 0.$
7. **Line Integral:** $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z)\, dx + (x - z)\, dy + (x - y)\, dz$
8. $\displaystyle\qquad = \int_C -y\, dx + x\, dy$
9. Letting $x = \cos t,$
10. $y = \sin t,$ we have
11. $dx = -\sin t\, dt,$
12. $dy = \cos t\, dt$ and
13. $\displaystyle\int_C -y\, dx + x\, dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$
14. $\qquad = 2\pi.$
15. **Double Integral:** Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Then
16. $\displaystyle\mathbf{N} = \frac{\nabla F}{\|\nabla F\|}$
17. $\displaystyle\qquad = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}}$
18. $\qquad = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$
19. Since $z^2 = 1 - x^2 - y^2,$
20. $\displaystyle z_x = \frac{-2x}{2z}$
21. $\displaystyle\qquad = \frac{-x}{z},$ and
22. $\displaystyle z_y = \frac{-y}{z},$
23. $\displaystyle dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA$
24. $\displaystyle\qquad = \frac{1}{z}\, dA.$
25. Now, since $\mathbf{curl\ F} = 2\mathbf{k}$, we have
26. $\displaystyle\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int 2z\left(\frac{1}{z}\right) dA$
27. $\displaystyle\qquad = \int_R\int 2\, dA$
28. $\qquad = 2(\text{Area of circle of radius } 1)$
29. $\qquad = 2\pi.$