MATH152 CALCULUS II TUTORIAL – 12

(29.12.2017)

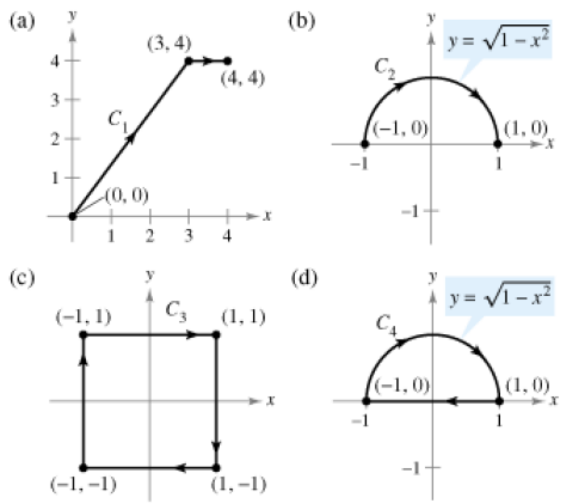
Question 1: Let $F(x,y) = y^2 i + 2xyj$

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If $\mathbf{F}$ is conservative, the integration may be easier on an alternative path.)

$$\int_C y^2\,dx + 2xy\,dy$$



1. Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative.
2. The potential function is $f(x, y) = xy^2 + k$.
3. Therefore, we can use the Fundamental Theorem of Line Integrals.
4. (a) $\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(0,\,0)}^{(4,\,4)}$
5. $= 64$
6. (b) $\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(-1,\,0)}^{(1,\,0)}$
7. $= 0$
8. (c) and (d) Since C is a closed curve,

$$\int_C y^2\,dx + 2xy\,dy = 0.$$

Question 2 :

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If $\mathbf{F}$ is conservative, the integration may be easier on an alternative path.)

$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 2$

1. Since $\mathbf{curl\ F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative.
2. The potential function is $f(x, y, z) = xyz + k$.
3. (a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$
4. $\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,\,2,\,0)}^{(4,\,2,\,4)}$
5. $= 32$

Question 3 :

Use Green's Theorem to evaluate the integral

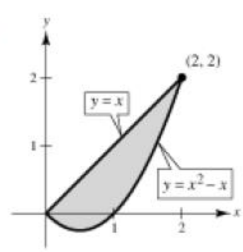
$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$

1.



2. $\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2 \int_{x^2 - x}^{x} dy\,dx$

3. $= \int_0^2 (2x - x^2)\,dx$

4. $= \frac{4}{3}$

Question 4:

Use Green's Theorem to evaluate the line integral.

$$\int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy$$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

1. By Green's Theorem,

$$\int_C \sin x \cos y\,dx + (xy + \cos x \sin y)\,dy = \int_R\int [(y - \sin x \sin y) - (-\sin x$$

2. $= \int_0^1 \int_x^{\sqrt{x}} y\,dy\,dx$

3. $= \frac{1}{2}\int_0^1 (x - x^2)\,dx$

4. $= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

5. $= \frac{1}{12}.$

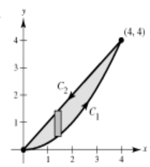
Question 5:

Verify Green's Theorem by evaluating both integrals

$$\int_C y^2\,dx + x^2\,dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

for the given path.

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2/4$

1. 

2. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8 - t)\mathbf{i} + (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

3. $\int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 [(8 - t)^2(-dt) + (8 - t)^2(-dt)]$

4. $= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8 - t)^2\,dt$

5. $= \frac{224}{5} - \frac{128}{3}$

6. $= \frac{32}{15}$

7. By Green's Theorem,

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4 \int_{x^2/4}^{x} (2x - 2y)\,dy\,dx$$

8. $= \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx$

9. $= \frac{32}{15}.$

Question : 6

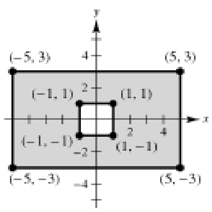
Use Green's Theorem to evaluate the integral

$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying inside the rectangle bounded by $x = -5$, $x = 5$, $y = -3$, and $y = 3$, and outside the square bounded by $x = -1$, $x = 1$, $y = -1$, and $y = 1$

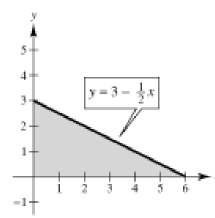
1. From the accompanying figure, we see that R is the shaded region.
2. 

3. Thus, Green's Theorem yields
4. $\int_C (y - x)\,dx + (2x - y)\,dy = \int\int_R 1\,dA$
5. $= \text{Area of } R$
6. $= 6(10) - 2(2)$
7. $= 56.$

Question : 7

Evaluate $\int\int_S xy\,dS.$

S: $z = 6 - x - 2y$, first octant

1. $\frac{\partial z}{\partial x} = -1$
2. $\frac{\partial z}{\partial y} = -2$
3. $\int\int_S xy\,dS = \int_0^6 \int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\,dy\,dx$
4. $= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$
5. $= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$
6. $= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6$
7. $= \frac{27\sqrt{6}}{2}$
8. 

Question : 8

Evaluate $\iint_S (x - 2y + z)\, dS$.

S: $z = 4 - x$, $0 \le x \le 4$, $0 \le y \le 4$

1. $\dfrac{\partial z}{\partial x} = -1$

2. $\dfrac{\partial z}{\partial y} = 0$

3. $\iint_S (x - 2y + z)\, dS = \int_0^4 \int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\, dy\, dx$

4. $\quad = \sqrt{2}\int_0^4 \int_0^4 (4 - 2y)\, dy\, dx$

5. $\quad = 0$

Question : 9

Find the flux of $\mathbf{F}$ through S,

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

where $\mathbf{N}$ is the upward unit normal vector to S.

$\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

S: $x + y + z = 1$, first octant

1. $G(x, y, z) = x + y + z - 1$

2. $\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

3. $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA$

4. $\quad = \int_0^1 \int_0^{1-x} (3z - 4 + y)\, dy\, dx$

5. $\quad = \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y]\, dy\, dx$

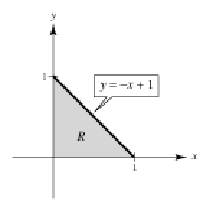
6. $\quad = \int_0^1 \int_0^{1-x} (-1 - 3x - 2y)\, dy\, dx$

7. $\quad = \int_0^1 \Big[-y - 3xy - y^2\Big]_0^{1-x} dx$

8. $\quad = -\int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2]\, dx$

9. $\quad = -\int_0^1 (2 - 2x^2)\, dx$

10. $\quad = -\dfrac{4}{3}$

11. 

Question : 10

Use the Divergence Theorem to evaluate

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS$$

and find the outward flux of $\mathbf{F}$ through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

S: $x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a$

1. Since div $\mathbf{F} = 2x + 2y + 2z$, we have
2. $\displaystyle \iiint_Q \text{div}\,\mathbf{F}\, dV = \int_0^a\int_0^a\int_0^a (2x + 2y + 2z)\, dz\, dy\, dx$
3. $\displaystyle = \int_0^a\int_0^a (2ax + 2ay + a^2)\, dy\, dx$
4. $\displaystyle = \int_0^a (2a^2x + 2a^3)\, dx$
5. $\displaystyle = \Big[a^2x^2 + 2a^3x\Big]_0^a$
6. $= 3a^4.$

Question : 11

Verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

$\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$

S: $z = \sqrt{1 - x^2 - y^2}$

1. In this case, $M = -y + z,$
2. $N = x - z,$
3. $P = x - y$ and
4. C is the circle $x^2 + y^2 = 1,$
5. $z = 0,$
6. $dz = 0.$
7. **Line Integral:** $\displaystyle \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z)\, dx + (x - z)\, dy + (x - y)\, dz$
8. $\displaystyle = \int_C -y\, dx + x\, dy$
9. Letting $x = \cos t,$
10. $y = \sin t,$ we have
11. $dx = -\sin t\, dt,$
12. $dy = \cos t\, dt$ and
13. $\displaystyle \int_C -y\, dx + x\, dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$
14. $= 2\pi.$
15. **Double Integral:** Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$. Then
16. $\displaystyle \mathbf{N} = \frac{\nabla F}{\|\nabla F\|}$
17. $\displaystyle = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}}$
18. $= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$
19. Since $z^2 = 1 - x^2 - y^2,$
20. $\displaystyle z_x = \frac{-2x}{2z}$
21. $\displaystyle = \frac{-x}{z},$ and
22. $\displaystyle z_y = \frac{-y}{z},$
23. $\displaystyle dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dA$
24. $\displaystyle = \frac{1}{z}\, dA.$
25. Now, since **curl F** $= 2\mathbf{k}$, we have
26. $\displaystyle \int_S\int (\textbf{curl F}) \cdot \mathbf{N}\, dS = \int_R\int 2z\left(\frac{1}{z}\right) dA$
27. $\displaystyle = \int_R\int 2\, dA$
28. $= 2$(Area of circle of radius 1)
29. $= 2\pi.$