MATH152 CALCULUS II TUTORIAL – 12

(08.01.2016)

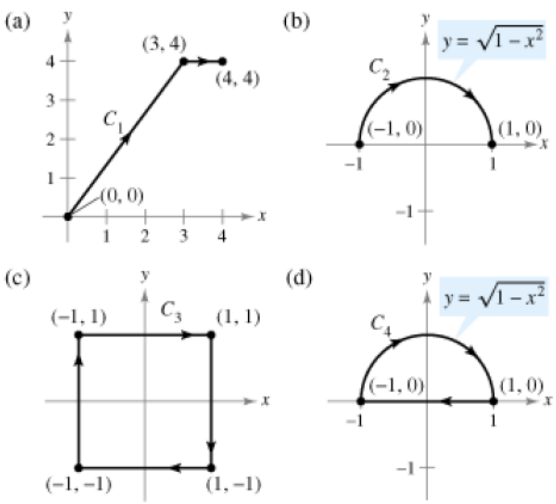
Question 1: Let $F(x,y) = y^2 i + 2xyj$

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If **F** is conservative, the integration may be easier on an alternative path.)

$$\int_C y^2\,dx + 2xy\,dy$$



1. Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative.
2. The potential function is $f(x, y) = xy^2 + k$.
3. Therefore, we can use the Fundamental Theorem of Line Integrals.
4. (a) $\displaystyle\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(0,0)}^{(4,4)}$
5. $= 64$
6. (b) $\displaystyle\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(-1,0)}^{(1,0)}$
7. $= 0$
8. (c) and (d) Since C is a closed curve,

$$\int_C y^2\,dx + 2xy\,dy = 0.$$

Question 2 :

Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If **F** is conservative, the integration may be easier on an alternative path.)

$\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 2$

1. Since $\mathbf{curl\ F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative.
2. The potential function is $f(x, y, z) = xyz + k$.
3. (a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 4$
4. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xyz\Big]_{(0,2,0)}^{(4,2,4)}$
5. $= 32$

Question 3 :

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and

$$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$$

as shown in Figure 15.20.

Solution

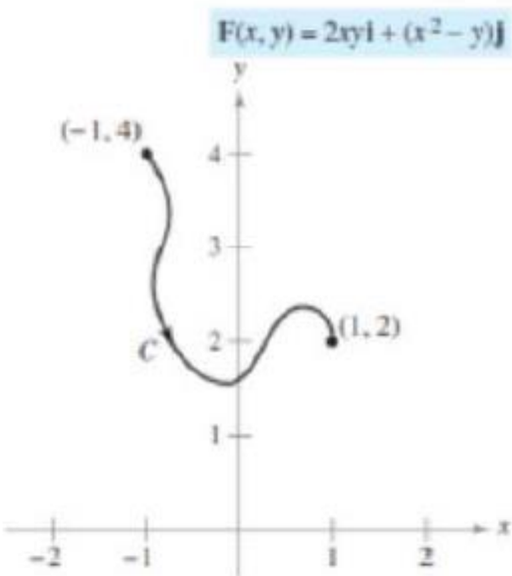
you know that **F** is the gradient of f where

$$f(x, y) = x^2y - \frac{y^2}{2} + K.$$

Consequently, **F** is conservative, and by the Fundamental Theorem of Line Integrals, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(1, 2) - f(-1, 4)\\ &= \left[1^2(2) - \frac{2^2}{2}\right] - \left[(-1)^2(4) - \frac{4^2}{2}\right]\\ &= 4.\end{aligned}$$

Note that it is unnecessary to include a constant K as part of f, because it is canceled by subtraction.



Using the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Figure 15.20

Question 4:

Evaluate the line integral using the Fundamental

Theorem of Line Integrals.

$$\int_C (y + 2z)\,dx + (x - 3z)\,dy + (2x - 3y)\,dz$$

(a) C: line segment from $(0, 0, 0)$ to $(1, 1, 1)$

(b) C: line segments from $(0, 0, 0)$ to $(0, 0, 1)$ to $(1, 1, 1)$

(c) C: line segments from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$

1. $\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

2. (a) $\left[xy - 3yz + 2xz\right]_{(0,0,0)}^{(1,1,1)} = 0 - 0$

3. $= 0$

4. (b) $\left[xy - 3yz + 2xz\right]_{(0,0,0)}^{(0,0,1)} + \left[xy - 3yz + 2xz\right]_{(0,0,1)}^{(1,1,1)} = 0 + 0$

5. $= 0$

6. (c) $\left[xy - 3yz + 2xz\right]_{(0,0,0)}^{(1,0,0)} + \left[xy - 3yz + 2xz\right]_{(1,0,0)}^{(1,1,0)} +$

$\left[xy - 3yz + 2xz\right]_{(1,1,0)}^{(1,1,1)} = 0 + 1 + (-1)$

7. $= 0$

Question 5:

Use Green's Theorem to evaluate the integral

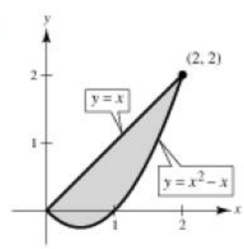
$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$

1.



2. $\displaystyle \int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2 \int_{x^2 - x}^{x} dy\,dx$

3. $\displaystyle = \int_0^2 (2x - x^2)\,dx$

4. $\displaystyle = \frac{4}{3}$

Question 6:

Use Green's Theorem to evaluate the line integral.

$$\int_C \sin x \cos y \, dx + (xy + \cos x \sin y) \, dy$$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

1. By Green's Theorem,

$$\int_C \sin x \cos y \, dx + (xy + \cos x \sin y) \, dy = \int\int_R [(y - \sin x \sin y) - (-\sin x$$

2. $$= \int_0^1 \int_x^{\sqrt{x}} y \, dy \, dx$$

3. $$= \frac{1}{2} \int_0^1 (x - x^2) \, dx$$

4. $$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

5. $$= \frac{1}{12}.$$

Question 7:

Evaluate

$$\int_C (\arctan x + y^2) \, dx + (e^y - x^2) \, dy$$

where C is the path enclosing the annular region shown in Figure 15.31.

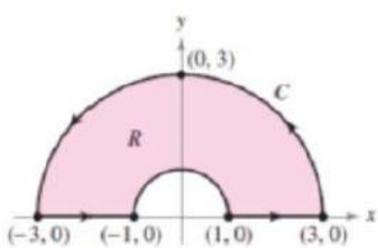
Solution

In polar coordinates, R is given by $1 \le r \le 3$ for $0 \le \theta \le \pi$. Moreov

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2x - 2y = -2(r \cos \theta + r \sin \theta).$$

So, by Green's Theorem,

$$\int_C (\arctan x + y^2) \, dx + (e^y - x^2) \, dy = \int\int_R -2(x + y) \, dA$$



C is piecewise smooth.

Figure 15.31

$$= \int_0^{\pi} \int_1^3 -2r(\cos \theta + \sin \theta) r \, dr \, d\theta$$

$$= \int_0^{\pi} -2(\cos \theta + \sin \theta) \frac{r^3}{3} \Big]_1^3 \, d\theta$$

$$= \int_0^{\pi} \left(-\frac{52}{3}\right)(\cos \theta + \sin \theta) \, d\theta$$

$$= -\frac{52}{3} \Big[\sin \theta - \cos \theta \Big]_0^{\pi}$$

$$= -\frac{104}{3}.$$

Question : 8

Line integral as a double integral Evaluate $\oint_C (4x^3 + \sin y^2) \, dy - (4y^3 + \cos x^2) \, dx$, where C is the boundary of the disk $R = \{(x, y) : x^2 + y^2 \le 4\}$ oriented counterclockwise.

Letting $f(x, y) = 4x^3 + \sin y^2$ and $g(x, y) = 4y^3 + \cos x^2$, Green's Theorem takes the form

$$\oint_C \underbrace{(4x^3 + \sin y^2)}_{f} \, dy - \underbrace{(4y^3 + \cos x^2)}_{g} \, dx = \iint_R (\underbrace{12x^2}_{f_x} + \underbrace{12y^2}_{g_y}) \, dA$$

$$= \iint_R (\underbrace{12x^2}_{f_x} + \underbrace{12y^2}_{g_y}) \, dA$$

$$= 12 \int_0^{2\pi} \int_0^2 r^2 \, r \, dr \, d\theta$$

$$= 12 \int_0^{2\pi} \frac{r^4}{4} \Big|_0^2 \, d\theta$$

$$= 48 \int_0^{2\pi} d\theta = 96\pi.$$

Question : 9

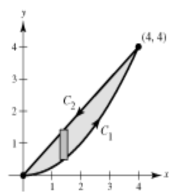
Verify Green's Theorem by evaluating both integrals

$$\int_C y^2\,dx + x^2\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

for the given path.

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2/4$

1.



2. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8 - t)\mathbf{i} + (8 - t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

3. $\int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 [(8 - t)^2(-dt) + (8 - t)^2(-d$

4. $= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8 - t)^2\,dt$

5. $= \frac{224}{5} - \frac{128}{3}$

6. $= \frac{32}{15}$

7. By Green's Theorem,

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_{x^2/4}^{x} (2x - 2y)\,dy\,dx$$

8. $= \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx$

9. $= \frac{32}{15}.$

Question : 10

Use Green's Theorem to evaluate the integral

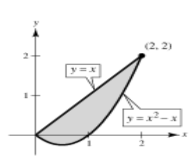
$$\int_C (y - x)\,dx + (2x - y)\,dy$$

for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - x$

1.



2. $\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2\int_{x^2-x}^{x} dy\,dx$

3. $= \int_0^2 (2x - x^2)\,dx$

4. $= \frac{4}{3}$

Question : 11

Use Green's Theorem to evaluate the integral

$$\int_C (y - x)\,dx + (2x - y)\,dy$$

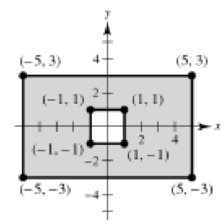
for the given path.

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1.$$

C: boundary of the region lying inside the rectangle bounded by $x = -5$, $x = 5$, $y = -3$, and $y = 3$, and outside the square bounded by $x = -1$, $x = 1$, $y = -1$, and $y = 1$

1. From the accompanying figure, we see that R is the shaded region.

2.



3. Thus, Green's Theorem yields

4. $\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\int 1\,dA$

5. $=$ Area of R

6. $= 6(10) - 2(2)$

7. $= 56.$

Question : 12

Use Green's Theorem to calculate the work done by the force $\mathbf{F}$ on a particle that is moving counterclockwise around the closed path C.

$\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

C: $x^2 + y^2 = 4$

1. Work $= \int_C xy\,dx + (x + y)\,dy$

2. $= \int_R\int (1 - x)\,dA$

3. $= \int_0^{2\pi}\int_0^2 (1 - r\cos\theta)r\,dr\,d\theta$

4. $= \int_0^{2\pi}\left(2 - \frac{8}{3}\cos\theta\right) d\theta$

5. $= 4\pi$

Question : 13

Use a line integral to find the area of the region R.

R: region bounded by the graph of $x^2 + y^2 = a^2$

1. C: let $x = a\cos t$,
2. $y = a\sin t,\ 0 \le t \le 2\pi$.
3. By Theorem 15.9, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

4. $= \frac{1}{2}\int_0^{2\pi} [a\cos t(a\cos t) - a\sin t(-a\sin t)]\,dt$
5. $= \frac{1}{2}\int_0^{2\pi} a^2\,dt$
6. $= \left[\frac{a^2}{2}t\right]_0^{2\pi}$
7. $= \pi a^2$.

Question : 14

Evaluate $\int\int_S (x - 2y + z)\,dS$.

S: $z = 4 - x,\ 0 \le x \le 4,\ 0 \le y \le 4$

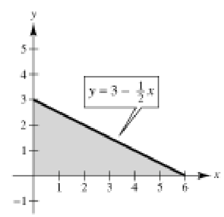
1. $\frac{\partial z}{\partial x} = -1$
2. $\frac{\partial z}{\partial y} = 0$
3. $\int\int_S (x - 2y + z)\,dS = \int_0^4\int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\,dy\,dx$
4. $= \sqrt{2}\int_0^4\int_0^4 (4 - 2y)\,dy\,dx$
5. $= 0$

Question : 15

Evaluate $\int\int_S xy\,dS$.

S: $z = 6 - x - 2y$, first octant

1. $\frac{\partial z}{\partial x} = -1$
2. $\frac{\partial z}{\partial y} = -2$
3. $\int\int_S xy\,dS = \int_0^6\int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\,dy\,dx$
4. $= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$
5. $= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$
6. $= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6$
7. $= \frac{27\sqrt{6}}{2}$
8.



Question : 16

Evaluate $\int\int_S f(x, y, z)\,dS$.

$f(x, y, z) = x^2 + y^2 + z^2$

S: $x^2 + y^2 = 9,\ 0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 9$

1. Project the solid onto the yz-plane; $x = \sqrt{9 - y^2},\ 0 \le y \le 3,\ 0 \le z \le 9$.
2. $\int\int_S f(x, y, z)\,dS = \int_0^3\int_0^9 [(9 - y^2) + y^2 + z^2]\sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 + (0)^2}\,dz\,dy$
3. $= \int_0^3\int_0^9 (9 + z^2)\frac{3}{\sqrt{9 - y^2}}\,dz\,dy$
4. $= \int_0^3 \left[\frac{3}{\sqrt{9 - y^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^9 dy$
5. $= 324\int_0^3 \frac{3}{\sqrt{9 - y^2}}\,dy$
6. $= \left[972\arcsin\left(\frac{y}{3}\right)\right]_0^3$
7. $= 972\left(\frac{\pi}{2} - 0\right)$
8. $= 486\pi$