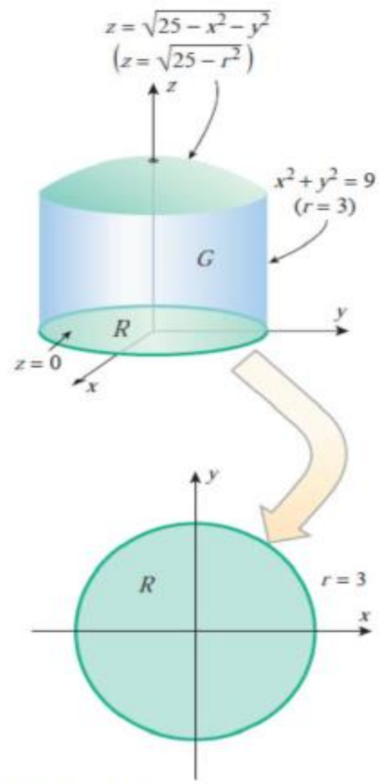
MATH152 CALCULUS II   TUTORIAL – 11

(12.05.2017)

Question 1:

Use triple integral in cylindrical coordinates to find the volume of the solid G that is bounded above by the semi-sphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.



▲ **Figure 14.6.5**

The solid G and its projection R on the xy-plane are shown in Figure 14.6.5.

In cylindrical coordinates, the upper surface of G is the hemisphere $z = \sqrt{25 - r^2}$ and the lower surface is the plane $z = 0$. Thus, from (4), the volume of G is

$$V = \iiint_G dV = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

For the double integral over R, we use polar coordinates:

$$V = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^3 \left[rz \right]_{z=0}^{\sqrt{25-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^3 r\sqrt{25-r^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25-r^2)^{3/2} \right]_{r=0}^{3} d\theta$$

$$u = 25 - r^2, \quad du = -2r\,dr$$

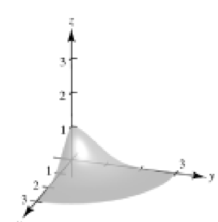
$$= \int_0^{2\pi} \frac{61}{3}\,d\theta = \frac{122}{3}\pi$$ ◀

Question 2 :

Sketch the solid region whose volume is given by the iterated integral, and evaluate the iterated integral.

$$\int_0^{\pi/2} \int_0^3 \int_0^{e^{-r^2}} r\,dz\,dr\,d\theta$$

1.



2. $\displaystyle \int_0^{\pi/2} \int_0^3 \int_0^{e^{-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^3 re^{-r^2}\,dr\,d\theta$

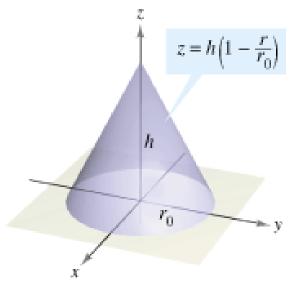
3. $\displaystyle = \int_0^{\pi/2} \left[-\frac{1}{2}e^{-r^2} \right]_0^3 d\theta$

4. $\displaystyle = \int_0^{\pi/2} \frac{1}{2}(1 - e^{-9})\,d\theta$

5. $\displaystyle = \frac{\pi}{4}(1 - e^{-9})$

Question 3 :

Use cylindrical coordinates to find the indicated characteristic of the cone shown in the figure.



Find the volume of the cone.

1. $z = h - \frac{h}{r_0}\sqrt{x^2 + y^2}$
2. $= \frac{h}{r_0}(r_0 - r)$
3. $V = 4\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0 - r)/r_0} r\,dz\,dr\,d\theta$
4. $= \frac{4h}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0 r - r^2)\,dr\,d\theta$
5. $= \frac{4h}{r_0}\int_0^{\pi/2}\frac{r_0^3}{6}\,d\theta$
6. $= \frac{4h}{r_0}\left(\frac{r_0^3}{6}\right)\left(\frac{\pi}{2}\right)$
7. $= \frac{1}{3}\pi r_0^2 h$

Question 4:

Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

1. $\frac{\partial}{\partial y}[2xy] = 2x$
2. $\frac{\partial}{\partial x}[x^2] = 2x$
3. Conservative
4. $f_x(x, y) = 2xy$
5. $f_y(x, y) = x^2$
6. $f(x, y) = x^2y + K$

Question 5:

Find the gradient vector field for the scalar function. (That is, find the conservative vector field for the potential function.)

$g(x, y, z) = xy\ln(x + y)$

1. $g_x(x, y, z) = y\ln(x + y) + \frac{xy}{x + y}$
2. $g_y(x, y, z) = x\ln(x + y) + \frac{xy}{x + y}$
3. $g_z(x, y, z) = 0$
4. $\mathbf{G}(x, y, z) = \left[\frac{xy}{x + y} + y\ln(x + y)\right]\mathbf{i} + \left[\frac{xy}{x + y} + x\ln(x + y)\right]\mathbf{j}$

Question 6:

Determine whether the vector field is conservative. If it is, find a potential function for the vector field.

$\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

1. $\frac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$
2. $\frac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$
3. Conservative
4. $f_x(x, y) = 2xye^{x^2y}$
5. $f_y(x, y) = x^2e^{x^2y}$
6. $f(x, y) = e^{x^2y} + K$

Question 7:

Determine whether the vector field $\mathbf{F}$ is conservative. If it is, find a potential function for the vector field.

$\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

1. $\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix}$
2. $= \mathbf{0}$
3. Conservative
4. $f_x(x, y, z) = ye^z$
5. $f_y(x, y, z) = xe^z$
6. $f_z(x, y, z) = xye^z$
7. $f(x, y, z) = xye^z + K$

Question : 8

Find **curl F** for the vector field at the given point.

Vector Field	*Point*
$\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	$(1, 2, 1)$

1. $\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix}$
2. $= xy\mathbf{j} - xz\mathbf{k}$
3. $\textbf{curl F}\,(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$

Question : 9

Determine whether the vector field **F** is conservative. If it is, find a potential function for the vector field.

$\mathbf{F}(x, y, z) = \sin y\mathbf{i} - x\cos y\mathbf{j} + \mathbf{k}$

1. $\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x\cos y & 1 \end{vmatrix}$
2. $= -2\cos y\mathbf{k}$
3. $\neq \mathbf{0}$
4. Not conservative

Question : 10

Find div $(\mathbf{F} \times \mathbf{G})$.

$\mathbf{F}(x, y, z) = \mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

1. $\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix}$
2. $= (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$
3. $\text{div}\,(\mathbf{F} \times \mathbf{G}) = 2z + 3x$

Question : 11

Find the divergence of the vector field **F**.

$\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + z^2\mathbf{k}$

1. $\text{div}\,\mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2]$
2. $= \cos x - \sin y + 2z$

Question : 12

Evaluate the line integral along the given path.

$\int_C (x - y)\, ds$

C: $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 2$

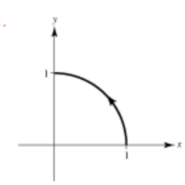
1. $\mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$
2. $\int_C (x - y)\, ds = \int_0^2 (4t - 3t)\sqrt{(4)^2 + (3)^2}\; dt$
3. $= \int_0^2 5t\, dt$
4. $= \left[\frac{5t^2}{2}\right]_0^2$
5. $= 10$

Question : 13

Evaluate $\int_C (x^2 + y^2)\, ds$ along the given path.

C: counterclockwise around the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

1.



2. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$
3. $\int_C (x^2 + y^2)\, ds = \int_0^{\pi/2} [\cos^2 t + \sin^2 t]\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$
4. $= \int_0^{\pi/2} dt$
5. $= \frac{\pi}{2}$

Question : 14

Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)$
$(x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \le t \le 1$, from (0
(Figure 16.18).

First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$$
$$= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.$$

Substitute $x = t$, $y = t^2, z = t^3$.

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = [(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$$
$$= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8$$

so,

$$\text{Work} = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt$$
$$= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.$$

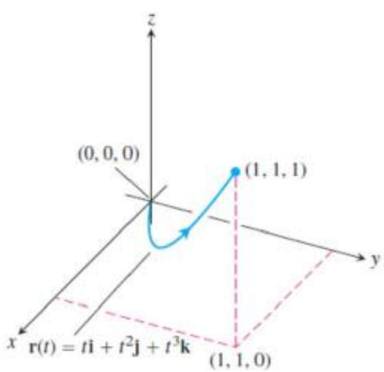


FIGURE 16.18

Question : 15

Let C be the circle of radius 3 given by

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}, \quad 0 \le t \le 2\pi$$

as shown in Figure 15.17. Evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy.$$

Because $x = 3\cos t$ and $y = 3\sin t$, you have
$dy = 3\cos t\,dt$. So, the line integral is

$$\int_C M\,dx + N\,dy$$
$$= \int_C y^3\,dx + (x^3 + 3xy^2)\,dy$$
$$= \int_0^{2\pi} [(27\sin^3 t)(-3\sin t) + (27\cos^3 t + 81\cos t$$
$$= 81\int_0^{2\pi} (\cos^4 t - \sin^4 t + 3\cos^2 t\sin^2 t)\,dt$$
$$= 81\int_0^{2\pi} \left(\cos^2 t - \sin^2 t + \frac{3}{4}\sin^2 2t\right) dt$$
$$= 81\int_0^{2\pi} \left[\cos 2t + \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right)\right] dt$$
$$= 81\left[\frac{\sin 2t}{2} + \frac{3}{8}t - \frac{3\sin 4t}{32}\right]_0^{2\pi}$$
$$= \frac{243\pi}{4}.$$

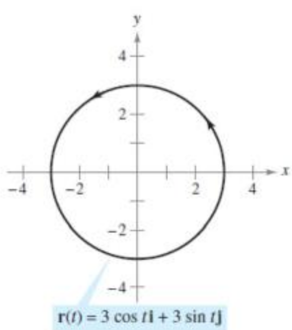


Figure 15.17

Question : 16

Evaluate

$$\int_C y\,dx + x^2\,dy$$

where C is the parabolic arc given by $y = 4x - x$
Figure 15.18.

Rather than converting to the parameter t, you
variable x and write

$$y = 4x - x^2 \quad \Rightarrow \quad dy = (4 - 2x)\,dx.$$

Then, in the direction from (4, 0) to (1, 3), the line int

$$\int_C y\,dx + x^2\,dy = \int_4^1 [(4x - x^2)\,dx + x^2(4 - 2x)$$

$$= \int_4^1 (4x + 3x^2 - 2x^3)\,dx$$

$$= \left[2x^2 + x^3 - \frac{x^4}{2}\right]_4^1 = \frac{69}{2}.$$