MATH152 CALCULUS II TUTORIAL – 10

(14.12.2018)

Question 1:

Use polar coordinates to set up and evaluate the double integral $\int_R\int f(x, y)\, dA$.

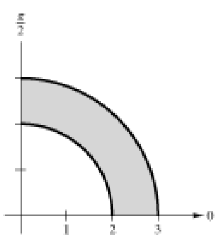
$f(x, y) = x + y$, R: $x^2 + y^2 \le 4$, $x \ge 0$, $y \ge 0$

1. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx = \int_0^{\pi/2}\int_0^2 (r\cos\theta + r\sin\theta) r\, dr\, d\theta$
2. $\displaystyle = \int_0^{\pi/2}\int_0^2 (\cos\theta + \sin\theta) r^2\, dr\, d\theta$
3. $\displaystyle = \frac{8}{3}\int_0^{\pi/2} (\cos\theta + \sin\theta)\, d\theta$
4. $\displaystyle = \left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{\pi/2}$
5. $\displaystyle = \frac{16}{3}$

Question 2 :

Evaluate the double integral $\int_R\int f(r, \theta)\, dA$, and sketch the region R.

$$\int_0^{\pi/2}\int_2^3 \sqrt{9 - r^2}\, r\, dr\, d\theta$$

1. $\displaystyle\int_0^{\pi/2}\int_2^3 \sqrt{9 - r^2}\, r\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}(9 - r^2)^{3/2}\right]_2^3 d\theta$
2. $\displaystyle = \left[\frac{5\sqrt{5}}{3}\theta\right]_0^{\pi/2}$
3. $\displaystyle = \frac{5\sqrt{5}\pi}{6}$
4. 

Question 3 :

Evaluate the iterated integral by converting to polar coordinates.

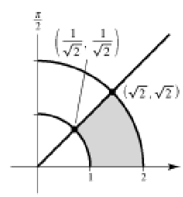
$$\int_0^3\int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx$$

1. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx = \int_0^{\pi/2}\int_0^3 r^4\, dr\, d\theta$
2. $\displaystyle = \frac{243}{5}\int_0^{\pi/2} d\theta$
3. $\displaystyle = \frac{243\pi}{10}$

Question 4:

Use polar coordinates to set up and evaluate the double integral $\int_R\int f(x, y)\, dA$.

$f(x, y) = \arctan\frac{y}{x}$, R: $x^2 + y^2 \ge 1$, $x^2 + y^2 \le 4$, $0 \le y \le x$

1. 

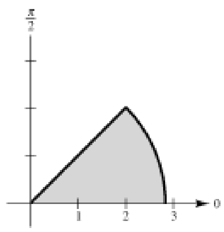
2. $\displaystyle\int_0^{1/\sqrt{2}}\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan\frac{y}{x}\, dx\, dy + \int_{1/\sqrt{2}}^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \arctan\frac{y}{x}\, dx\, dy = \int_0^{\pi/4}\int_1^2 \theta r\, dr\, d\theta$
3. $\displaystyle = \int_0^{\pi/4} \frac{3}{2}\theta\, d\theta$
4. $\displaystyle = \left[\frac{3\theta^2}{4}\right]_0^{\pi/4}$
5. $\displaystyle = \frac{3\pi^2}{64}$

Question 5:

Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

$$\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

1.



2. $$\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

$$= \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3}\,d\theta$$

$$= \frac{4\sqrt{2}\pi}{3}$$

Question 6:

Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.

$z = xy,\ x^2 + y^2 = 1,$ first octant

1. $V = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$
2. $= \frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3 \sin 2\theta\,dr\,d\theta$
3. $= \frac{1}{8}\int_0^{\pi/2} \sin 2\theta\,d\theta$
4. $= \left[-\frac{1}{16}\cos 2\theta\right]_0^{\pi/2}$
5. $= \frac{1}{8}$

Question 7:

Evaluate the triple integral.

$$\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz$$

1. $\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz = \int_1^4\int_0^1 \left[(2ze^{-x^2})y\right]_0^x dx\,dz$
2. $= \int_1^4\int_0^1 2zxe^{-x^2}\,dx\,dz$
3. $= \int_1^4 \left[-ze^{-x^2}\right]_0^1 dz$
4. $= \int_1^4 z(1-e^{-1})\,dz$
5. $= \left[(1-e^{-1})\frac{z^2}{2}\right]_1^4$
6. $= \frac{15}{2}\left(1-\frac{1}{e}\right)$

Question 8:

Evaluate the iterated integral.

$$\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx$$

1. $\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx = \int_0^1\int_0^x \left[xz\right]_0^{xy} dy\,dx$
2. $= \int_0^1\int_0^x x^2y\,dy\,dx$
3. $= \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^x dx$
4. $= \int_0^1 \frac{x^4}{2}\,dx$
5. $= \left[\frac{x^5}{10}\right]_0^1$
6. $= \frac{1}{10}$

Question : 9

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ a
between the vertical planes $x + y = 1$ and $x + y = 3$

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by th
$x^2 + y^2 + z^2 = 6$

a. In Figure 14.57, note that the solid is bounded below by the xy-plane ($z = ($
above by the cylinder $z = 1 - y^2$. So,

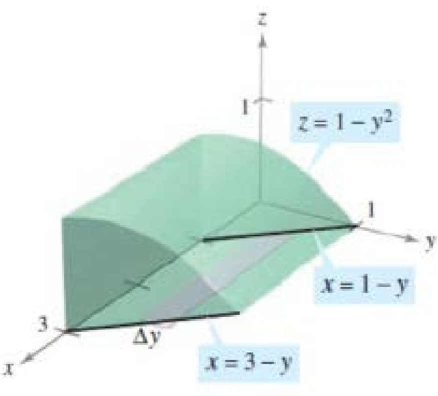
$$0 \le z \le 1 - y^2. \qquad \text{Bounds for } z$$

Projecting the region onto the xy-plane produces a parallelogram. Becaus
sides of the parallelogram are parallel to the x-axis, you have the following bc

$$1 - y \le x \le 3 - y \quad \text{and} \quad 0 \le y \le 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz\, dx\, dy.$$



Q: $0 \le z \le 1 - y^2$
$1 - y \le x \le 3 - y$
$0 \le y \le 1$

Figure 14.57

b. For the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$, you have

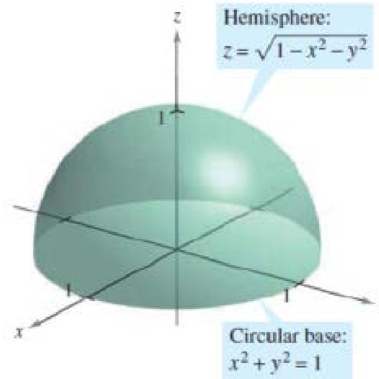
$$0 \le z \le \sqrt{1 - x^2 - y^2}. \qquad \text{Bounds for } z$$

In Figure 14.58, note that the projection of the hemisphere onto the xy-plane is the circle given by $x^2 + y^2 = 1$, and you can use either order $dx\,dy$ or $dy\,dx$. Choosing the first produces

$$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2} \quad \text{and} \quad -1 \le y \le 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\, dx\, dy.$$



Q: $0 \le z \le \sqrt{1 - x^2 - y^2}$
$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$
$-1 \le y \le 1$

Figure 14.58

c. For the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$, you have

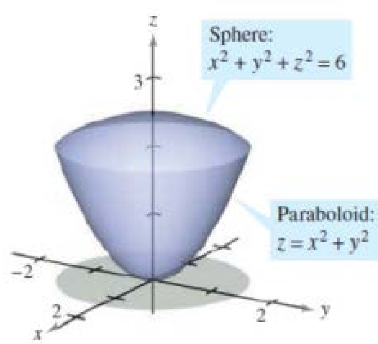
$$x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}. \qquad \text{Bounds for } z$$

The sphere and the paraboloid intersect at $z = 2$. Moreover, you can see in Figure 14.59 that the projection of the solid region onto the xy-plane is the circle given by $x^2 + y^2 = 2$. Using the order $dy\,dx$ produces

$$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \le x \le \sqrt{2}$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz\, dy\, dx.$$ ■



Q: $x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}$
$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2}$
$-\sqrt{2} \le x \le \sqrt{2}$

Figure 14.59

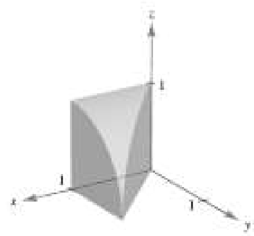
Question : 10

Sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

$$\int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} dz\, dx\, dy$$

Rewrite using the order $dz\, dy\, dx$.

1.



2. Top cylinder: $y^2 + z^2 = 1$

3. Side plane: $x = y$

4. $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\, dy\, dx$