

# MATH152 CALCULUS II TUTORIAL – 10

(15.12.2017)

## Question 1:

Use polar coordinates to set up and evaluate the double integral  $\int_R \int f(x, y) dA$ .

$$f(x, y) = x + y, R: x^2 + y^2 \leq 4, x \geq 0, y \geq 0$$

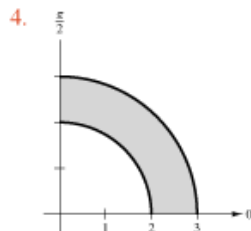
1.  $\int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) dy dx = \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r dr d\theta$
2.  $= \int_0^{\pi/2} \int_0^2 (\cos \theta + \sin \theta) r^2 dr d\theta$
3.  $= \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) d\theta$
4.  $= \left[ \frac{8}{3} (\sin \theta - \cos \theta) \right]_0^{\pi/2}$
5.  $= \frac{16}{3}$

## Question 2 :

Evaluate the double integral  $\int_R \int f(r, \theta) dA$ , and sketch the region  $R$ .

$$\int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} r dr d\theta$$

1.  $\int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} r dr d\theta = \int_0^{\pi/2} \left[ -\frac{1}{3}(9-r^2)^{3/2} \right]_2^3 d\theta$
2.  $= \left[ \frac{5\sqrt{5}}{3} \theta \right]_0^{\pi/2}$
3.  $= \frac{5\sqrt{5}\pi}{6}$



## Question 3 :

Evaluate the iterated integral by converting to polar coordinates.

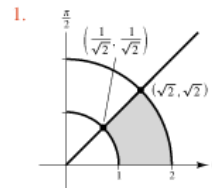
$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} dy dx$$

1.  $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} dy dx = \int_0^{\pi/2} \int_0^3 r^4 dr d\theta$
2.  $= \frac{243}{5} \int_0^{\pi/2} d\theta$
3.  $= \frac{243\pi}{10}$

## Question 4:

Use polar coordinates to set up and evaluate the double integral  $\int_R \int f(x, y) dA$ .

$$f(x, y) = \arctan \frac{y}{x}, R: x^2 + y^2 \geq 1, x^2 + y^2 \leq 4, 0 \leq y \leq x$$

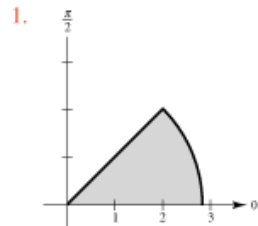


2.  $\int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy = \int_0^{\pi/4} \int_1^2 \theta r dr d\theta$
3.  $= \int_0^{\pi/4} \frac{3}{2} \theta d\theta$
4.  $= \left[ \frac{3\theta^2}{4} \right]_0^{\pi/4}$
5.  $= \frac{3\pi^2}{64}$

### Question 5:

Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

$$\int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} dy dx$$



2. 
$$\int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} dy dx$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 dr d\theta$$

$$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3} d\theta$$

$$= \frac{4\sqrt{2}\pi}{3}$$

### Question 6:

Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.

$$z = xy, \quad x^2 + y^2 = 1, \quad \text{first octant}$$

1. 
$$V = \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta)r dr d\theta$$

2. 
$$= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta dr d\theta$$

3. 
$$= \frac{1}{8} \int_0^{\pi/2} \sin 2\theta d\theta$$

4. 
$$= \left[ -\frac{1}{16} \cos 2\theta \right]_0^{\pi/2}$$

5. 
$$= \frac{1}{8}$$

### Question 7:

Evaluate the triple integral.

$$\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz$$

1. 
$$\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz = \int_1^4 \int_0^1 \left[ (2ze^{-x^2})y \right]_0^x dx dz$$

2. 
$$= \int_1^4 \int_0^1 2zxe^{-x^2} dx dz$$

3. 
$$= \int_1^4 \left[ -ze^{-x^2} \right]_0^1 dz$$

4. 
$$= \int_1^4 z(1 - e^{-1}) dz$$

5. 
$$= \left[ (1 - e^{-1}) \frac{z^2}{2} \right]_1^4$$

6. 
$$= \frac{15}{2} \left( 1 - \frac{1}{e} \right)$$

### Question 8:

Evaluate the iterated integral.

$$\int_0^1 \int_0^x \int_0^{xy} x dz dy dx$$

1. 
$$\int_0^1 \int_0^x \int_0^{xy} x dz dy dx = \int_0^1 \int_0^x \left[ xz \right]_0^{xy} dy dx$$

2. 
$$= \int_0^1 \int_0^x x^2 y dy dx$$

3. 
$$= \int_0^1 \left[ \frac{x^2 y^2}{2} \right]_0^x dx$$

4. 
$$= \int_0^1 \frac{x^4}{2} dx$$

5. 
$$= \left[ \frac{x^5}{10} \right]_0^1$$

6. 
$$= \frac{1}{10}$$

**Question : 9**

Set up a triple integral for the volume of each solid region.

- a. The region in the first octant bounded above by the cylinder  $z = 1 - y^2$ ; between the vertical planes  $x + y = 1$  and  $x + y = 3$
- b. The upper hemisphere given by  $z = \sqrt{1 - x^2 - y^2}$
- c. The region bounded below by the paraboloid  $z = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 6$

a. In Figure 14.57, note that the solid is bounded below by the  $xy$ -plane ( $z = 0$ ) and above by the cylinder  $z = 1 - y^2$ . So,

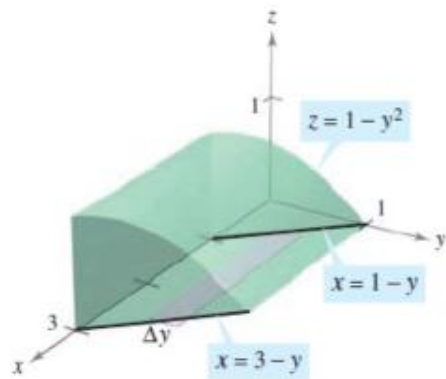
$$0 \leq z \leq 1 - y^2. \quad \text{Bounds for } z$$

Projecting the region onto the  $xy$ -plane produces a parallelogram. Because sides of the parallelogram are parallel to the  $x$ -axis, you have the following bounds:

$$1 - y \leq x \leq 3 - y \quad \text{and} \quad 0 \leq y \leq 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz \, dx \, dy.$$



$$Q: \begin{aligned} 0 &\leq z \leq 1 - y^2 \\ 1 - y &\leq x \leq 3 - y \\ 0 &\leq y \leq 1 \end{aligned}$$

**Figure 14.57**

b. For the upper hemisphere given by  $z = \sqrt{1 - x^2 - y^2}$ , you have

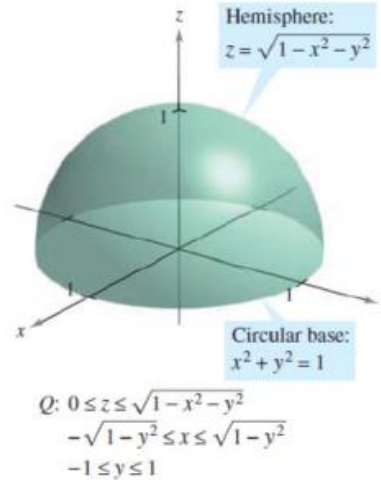
$$0 \leq z \leq \sqrt{1 - x^2 - y^2}. \quad \text{Bounds for } z$$

In Figure 14.58, note that the projection of the hemisphere onto the  $xy$ -plane is the circle given by  $x^2 + y^2 = 1$ , and you can use either order  $dx \, dy$  or  $dy \, dx$ . Choosing the first produces

$$-\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2} \quad \text{and} \quad -1 \leq y \leq 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \, dx \, dy.$$



**Figure 14.58**

c. For the region bounded below by the paraboloid  $z = x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 6$ , you have

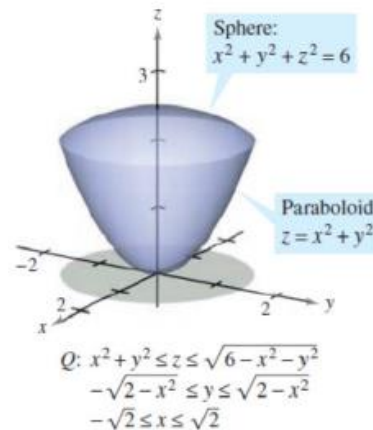
$$x^2 + y^2 \leq z \leq \sqrt{6 - x^2 - y^2}. \quad \text{Bounds for } z$$

The sphere and the paraboloid intersect at  $z = 2$ . Moreover, you can see in Figure 14.59 that the projection of the solid region onto the  $xy$ -plane is the circle given by  $x^2 + y^2 = 2$ . Using the order  $dy \, dx$  produces

$$-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \leq x \leq \sqrt{2}$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz \, dy \, dx.$$



**Figure 14.59**

**Question : 10**

Sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

$$\int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} dz dx dy$$

Rewrite using the order  $dz dy dx$ .

1.



2. Top cylinder:  $y^2 + z^2 = 1$

3. Side plane:  $x = y$

4. 
$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz dy dx$$