

MATH152 CALCULUS II TUTORIAL – 10

(16.12.2016)

Question 1:

Use polar coordinates to set up and evaluate the double integral $\int_R f(x, y) dA$.

$$f(x, y) = x + y, R: x^2 + y^2 \leq 4, x \geq 0, y \geq 0$$

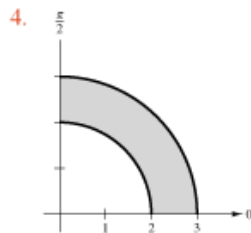
1. $\int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) dy dx = \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r dr d\theta$
2. $= \int_0^{\pi/2} \int_0^2 (\cos \theta + \sin \theta) r^2 dr d\theta$
3. $= \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) d\theta$
4. $= \left[\frac{8}{3} (\sin \theta - \cos \theta) \right]_0^{\pi/2}$
5. $= \frac{16}{3}$

Question 2 :

Evaluate the double integral $\int_R f(r, \theta) dA$, and sketch the region R .

$$\int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} r dr d\theta$$

1. $\int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} r dr d\theta = \int_0^{\pi/2} \left[-\frac{1}{3}(9-r^2)^{3/2} \right]_2^3 d\theta$
2. $= \left[\frac{5\sqrt{5}}{3} \theta \right]_0^{\pi/2}$
3. $= \frac{5\sqrt{5}\pi}{6}$



Question 3 :

Evaluate the iterated integral by converting to polar coordinates.

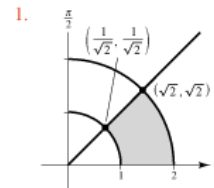
$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} dy dx$$

1. $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} dy dx = \int_0^{\pi/2} \int_0^3 r^4 dr d\theta$
2. $= \frac{243}{5} \int_0^{\pi/2} d\theta$
3. $= \frac{243\pi}{10}$

Question 4:

Use polar coordinates to set up and evaluate the double integral $\int_R f(x, y) dA$.

$$f(x, y) = \arctan \frac{y}{x}, R: x^2 + y^2 \geq 1, x^2 + y^2 \leq 4, 0 \leq y \leq x$$

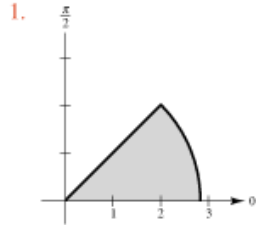


2. $\int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan \frac{y}{x} dx dy = \int_0^{\pi/4} \int_1^2 \theta r dr d\theta$
3. $= \int_0^{\pi/4} \frac{3}{2} \theta d\theta$
4. $= \left[\frac{3\theta^2}{4} \right]_0^{\pi/4}$
5. $= \frac{3\pi^2}{64}$

Question 5:

Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

$$\int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} dy dx$$



2.
$$\int_0^2 \int_0^x \sqrt{x^2 + y^2} dy dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} dy dx$$

$$= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 dr d\theta$$

$$= \int_0^{\pi/4} \frac{16\sqrt{2}}{3} d\theta$$

$$= \frac{4\sqrt{2}\pi}{3}$$

Question 6:

Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.

$$z = xy, \quad x^2 + y^2 = 1, \quad \text{first octant}$$

1.
$$V = \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta)r dr d\theta$$

2.
$$= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta dr d\theta$$

3.
$$= \frac{1}{8} \int_0^{\pi/2} \sin 2\theta d\theta$$

4.
$$= \left[-\frac{1}{16} \cos 2\theta \right]_0^{\pi/2}$$

5.
$$= \frac{1}{8}$$

Question 7:

Evaluate the triple integral.

$$\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz$$

1.
$$\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} dy dx dz = \int_1^4 \int_0^1 \left[(2ze^{-x^2})y \right]_0^x dx dz$$

2.
$$= \int_1^4 \int_0^1 2zxe^{-x^2} dx dz$$

3.
$$= \int_1^4 \left[-ze^{-x^2} \right]_0^1 dz$$

4.
$$= \int_1^4 z(1 - e^{-1}) dz$$

5.
$$= \left[(1 - e^{-1}) \frac{z^2}{2} \right]_1^4$$

6.
$$= \frac{15}{2} \left(1 - \frac{1}{e} \right)$$

Question 8:

Evaluate the iterated integral.

$$\int_0^1 \int_0^x \int_0^{xy} x dz dy dx$$

1.
$$\int_0^1 \int_0^x \int_0^{xy} x dz dy dx = \int_0^1 \int_0^x \left[xz \right]_0^{xy} dy dx$$

2.
$$= \int_0^1 \int_0^x x^2 y dy dx$$

3.
$$= \int_0^1 \left[\frac{x^2 y^2}{2} \right]_0^x dx$$

4.
$$= \int_0^1 \frac{x^4}{2} dx$$

5.
$$= \left[\frac{x^5}{10} \right]_0^1$$

6.
$$= \frac{1}{10}$$

Question : 9

Set up a triple integral for the volume of each solid region.

- a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$; between the vertical planes $x + y = 1$ and $x + y = 3$
- b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$
- c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$

a. In Figure 14.57, note that the solid is bounded below by the xy -plane ($z = 0$) and above by the cylinder $z = 1 - y^2$. So,

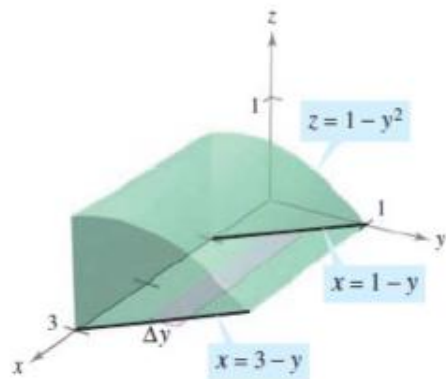
$$0 \leq z \leq 1 - y^2. \quad \text{Bounds for } z$$

Projecting the region onto the xy -plane produces a parallelogram. Because sides of the parallelogram are parallel to the x -axis, you have the following bounds:

$$1 - y \leq x \leq 3 - y \quad \text{and} \quad 0 \leq y \leq 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz \, dx \, dy.$$



$$Q: \begin{aligned} 0 &\leq z \leq 1 - y^2 \\ 1 - y &\leq x \leq 3 - y \\ 0 &\leq y \leq 1 \end{aligned}$$

Figure 14.57

b. For the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$, you have

$$0 \leq z \leq \sqrt{1 - x^2 - y^2}. \quad \text{Bounds for } z$$

In Figure 14.58, note that the projection of the hemisphere onto the xy -plane is the circle given by $x^2 + y^2 = 1$, and you can use either order $dx \, dy$ or $dy \, dx$. Choosing the first produces

$$-\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2} \quad \text{and} \quad -1 \leq y \leq 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \, dx \, dy.$$

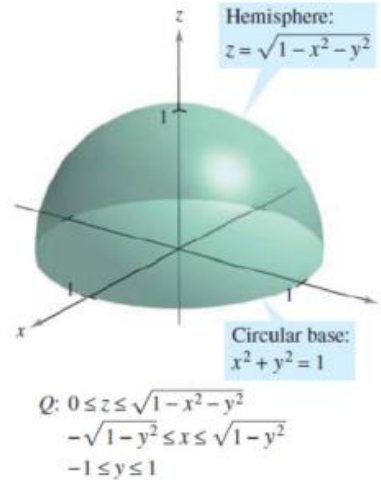


Figure 14.58

c. For the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$, you have

$$x^2 + y^2 \leq z \leq \sqrt{6 - x^2 - y^2}. \quad \text{Bounds for } z$$

The sphere and the paraboloid intersect at $z = 2$. Moreover, you can see in Figure 14.59 that the projection of the solid region onto the xy -plane is the circle given by $x^2 + y^2 = 2$. Using the order $dy \, dx$ produces

$$-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \leq x \leq \sqrt{2}$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz \, dy \, dx.$$

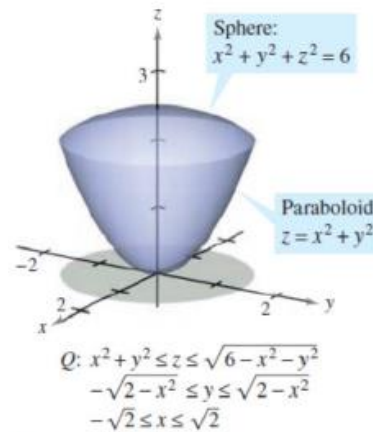


Figure 14.59

Question : 10

Sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

$$\int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} dz dx dy$$

Rewrite using the order $dz dy dx$.

1.



2. Top cylinder: $y^2 + z^2 = 1$

3. Side plane: $x = y$

4.
$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz dy dx$$