CALCULUS II

MIDTERM II EXAM SOLUTION

FALL 2016-2017

EMU
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS

Faculty of Arts and Sciences
Department of Mathematics
MATH152 Midterm II

Std. No: **Name :** **Surname :** **19.12.2016**

1	2	3	4	5	6	TOTAL

Duration : 90 minutes

Q-1) Investigate the following limits

a) (10 Points) $\lim_{(x,y)\to(0,1)} \frac{2xy+x^2y^2}{y^2x+2xy}$

$$\lim_{(x,y)\to(0,1)} \frac{2xy+x^2y^2}{y^2x+2xy} = \lim_{(x,y)\to(0,1)} \frac{xy(2+xy)}{xy(y+2)}$$
$$= \lim_{(x,y)\to(0,1)} \frac{2+xy}{y+2} = \frac{2}{3}$$

b) (10 Points) $\lim_{(x,y)\to(0,0)} \frac{x^3y+x^2y^2}{y^4+2x^4}$

Let $y = mx$ then $\lim_{(x,y)\to(0,0)} \frac{x^3y+x^2y^2}{y^4+2x^4} = \lim_{x\to 0} \frac{x^4(m+m^2)}{x^4(m^4+2)} = \lim_{x\to 0} \frac{m+m^2}{m^4+2}$

On the line $y = x$ $(m=1)$ limit is $\frac{2}{3}$

On the line $y = 2x$ $(m=2)$ limit is $\frac{6}{18} = \frac{1}{3}$

Different limits $\Rightarrow$ As a consequence of two-path rule limit does not exist

Q-2) **a)** (20 Points) Find the equation of tangent plane and normal line to the surface $z(x^2+y^2)=4$ at the point $P(1,1,2)$.

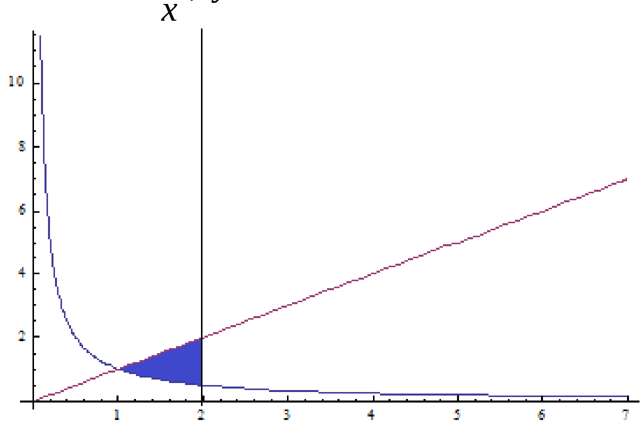
$f(x,y,z)=z(x^2+y^2)-4=0 \Rightarrow \vec{\nabla}f = 2xz\vec{i}+2yz\vec{j}+\left(x^2+y^2\right)\vec{k}$

$\Rightarrow \vec{\nabla}f(1,1,2)=\langle 4,4,2\rangle$

Tangent plane: $4(x-1)+4(y-1)+2(z-2)=0 \Rightarrow 2x+2y+z=6$

Normal line: $\begin{cases} x=1+4t \\ y=1+4t, \quad -\infty<t<\infty \\ z=2+2t \end{cases}$

Q-3) Use double integral to find the area of the region bounded by the graphs of the equations $y=\dfrac{1}{x}$, $y=x$ and $x=2$.



Intersection point $\Rightarrow \dfrac{1}{x}=x \Rightarrow x=1$

$$\text{Area}=\iint_Q 1\,dA=\int_1^2\int_{\frac{1}{x}}^{x} 1\,dy\,dx=\int_1^2\left(x-\frac{1}{x}\right)dx=\left(\frac{x^2}{2}-\ln x\right)\Bigg|_1^2=\frac{3}{2}-\ln 2$$

Q-4) If $f(x,y,z)=\dfrac{x}{y}-2z^2$, then find

a) (10 Points) Directional derivative of f at the point $P(1,1,2)$ in the direction of $\vec{u}=\langle 2,-1,2\rangle$

$$\frac{\partial f}{\partial x}=\frac{1}{y},\ \frac{\partial f}{\partial y}=\frac{-x}{y^2},\frac{\partial f}{\partial z}=-4z$$

$$\vec{\nabla}f\Big|_{P(1,1,2)}=\left\langle \frac{1}{y},\frac{-x}{y^2},-4z\right\rangle\Big|_{P(1,1,2)}=\langle 1,-1,-8\rangle$$

$$\|\vec{u}\|=\sqrt{4+1+4}=3\Rightarrow \vec{a}=\frac{\vec{u}}{\|\vec{u}\|}=\left\langle \frac{2}{3},\frac{-1}{3},\frac{2}{3}\right\rangle$$

$$\left(D_{\vec{a}}f\right)\Big|_P=\vec{\nabla}f\Big|_{P(1,1,2)}.\vec{a}=\langle 1,-1,-8\rangle.\left\langle \frac{2}{3},\frac{-1}{3},\frac{2}{3}\right\rangle=\frac{2}{3}+\frac{1}{3}-\frac{16}{3}=\frac{-13}{3}$$

b) (10 Points) In which direction f has minimal directional derivative at $P(1,1,2)$ and find the value of minimal directional derivative of f in that direction.

$-\vec{\nabla}f\Big|_{P(1,1,2)}=\langle -1,1,8\rangle$ direction of minimal directional derivative

$-\left\|\vec{\nabla}f\right\|=-\sqrt{1+1+64}=-\sqrt{66}$ value of the minimal directional derivative

Q-5) Use polar coordinates to evaluate $\iint_R y\left(x^2+y^2\right)dA$ where R is the upper half of the unit disc with boundary circle $x^2+y^2=1$.

$$R=\left\{(r,\theta):\ 0\le r\le 1,\ 0\le\theta\le\pi\right\},\ \ x=r\cos\theta,\ y=r\sin\theta\Rightarrow$$

$$\iint_R y\left(x^2+y^2\right)dA=\int_0^{x}\int_0^{1} r\sin\theta r^2 r dr d\theta=\int_0^{\pi}\sin\theta\left(\frac{r^5}{5}\Big|_0^1\right)d\theta=\frac{-1}{5}\left(\cos\theta|_0^{\pi}\right)$$

$$=\frac{-1}{5}\left(\cos\pi-\cos 0\right)=\frac{2}{5}$$

Q-6) Find extrema and saddle points of $f(x,y)=x^3+y^3+3x^2-3y^2+3$

$f_x=3x^2+6x=0\Rightarrow 3x(x+2)=0\Rightarrow x=0$ and $x=-2$

$f_y=3y^2-6y=0\Rightarrow 3y(y-2)\Rightarrow y=0,$ and $y=2$

Critical points are $(0,0),(0,2),(-2,0),(-2,2)$

$$\left.\begin{array}{l} f_{xx}=6x+6 \\ f_{yy}=6y-6 \\ f_{xy}=0 \end{array}\right\}\Rightarrow D(x,y)=(6x+6)(2y-6)$$

Critical Points	The value of $D(x,y)$	The value of f_{xx}	Classification
(0,0)	$D(0,0)=-36<0$	Irrelevant	Saddle point
(0,2)	$D(0,2)=36>0$	$f_{xx}(0,2)=6>0$	Local minimum
(-2,0)	$D(-2,0)=36>0$	$f_{xx}(-2,0)=-6<0$	Local maximum
(-2,2)	$D(-2,2)=-36<0$	Irrelevant	Saddle point