Faculty of Arts and Sciences
Department of Mathematics
MATH152 Midterm II

Std. No: **Name :** **Surname :** **09.05.2018**

1	2	3	4	5	6	TOTAL

Duration : 90 mins

Q-1) Given $f(x,y,z)=x^2-y^3+2xz^2$ and a point $P(2,1,-1)$ then

a) (6 points) Find $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} f(2,1,-1)$.

$\vec{\nabla} f(x,y,z)=(2x+2z^2)\,i-3y^2\,j+4xz\,k$

$\vec{\nabla} f(2,1,-1)=6i-3j-8k$

b) (6 points) Find the directional derivative of $f(x,y,z)$ at $P(2,1,-1)$ in the direction of $u=\left\langle 1,-1,\sqrt{2}\right\rangle$.

$$\frac{\vec{u}}{|\vec{u}|}=\frac{\langle 1,-1,\sqrt{2}\rangle}{2} \rightarrow D_u f(2,1,-1)=\vec{\nabla} f(2,1,-1)\cdot\frac{\vec{u}}{|\vec{u}|}$$

$$=\langle 6,-3,-8\rangle\cdot\frac{\langle 1,-1,\sqrt{2}\rangle}{2}=\frac{6+3-8\sqrt{2}}{2}=\frac{9-8\sqrt{2}}{2}$$

c) (4 points) Find the maximum directional derivative of $f(x,y,z)$ at $P(2,1,-1)$.

Maximum Directional derivative at $P(2,1,-1)$

$=\left|\vec{\nabla} f(2,1,-1)\right|=|6i-3j-8k|=\sqrt{36+9+64}=\sqrt{109}$

d) (4 points) In which direction $f(x,y,z)$ has minimum directional derivative at $P(2,1,-1)$.

$f(x,y,z)$ has minimum directional derivative at $P(2,1,-1)$ in the direction of

$-\vec{\nabla} f(2,1,-1)=-6i+3j+8k$

Q-2) Given $\int_0^1 \int_y^1 f(x,y)dxdy$

a) (10 Points) Reverse the order of integration.

Region R : $\begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$ or $\begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases} \Rightarrow \int_0^1 \int_y^1 f(x,y)dxdy = \int_0^1 \int_0^x f(x,y)dydx$

b) (10 Points) Evaluate the given integral for $f(x,y) = \sqrt{x^2+1}$

$$\int_0^1 \int_0^x f(x,y)dydx = \int_0^1 \int_0^x \sqrt{x^2+1}\, dydx = \int_0^1 x\sqrt{x^2+1}\ dx = \frac{1}{2}\int_1^2 u^{\frac{1}{2}}du = \frac{1}{3}u^{\frac{3}{2}} = \frac{\sqrt{8}-1}{3}$$

Q-3) (20 Points) Evaluate the area of the region bounded by the graphs of the equations $y = x^2$, $y = 2 - x$ and $x -$ axis.

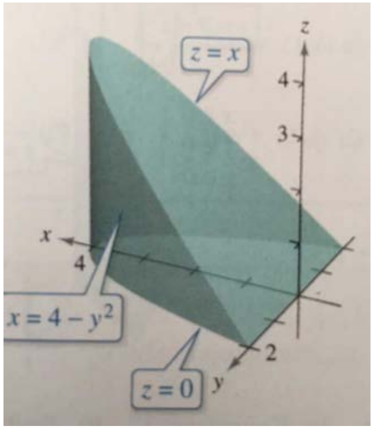
Area of the region R $= \iint_R 1 dA = \int_0^1 \int_{\sqrt{y}}^{2-y} 1 dxdy = \int_0^1 (2 - y - \sqrt{y})\, dy = 2y - \frac{y^2}{2} - \frac{2}{3} y^{\frac{3}{2}} = \frac{5}{6}$.

Or

Area of the region R $= \iint_R 1 dA = \int_0^1 \int_0^{x^2} 1 dydx + \int_1^2 \int_0^{2-x} 1 dydx = \frac{5}{6}$.

Q-4) Use triple integral to find the volume of the region bounded by $z = x,\ z = 0$ and $x = 4 - y^2$. (see the figure)

Solution :



$$Q \equiv \left\{ \begin{array}{c} -2 \le y \le 2 \\ 0 \le x \le 4 - y^2 \\ 0 \le z \le x \end{array} \right\} \Rightarrow$$

The Volume of the region $= \iiint_Q 1\,dA = \int_{-2}^{2} \int_{0}^{4-y^2} \int_{0}^{x} dz\,dx\,dy = \int_{-2}^{2} \int_{0}^{4-y^2} x\,dx\,dy$.

$$= \int_{-2}^{2} \frac{x^2}{2} \Big|_0^{4-y^2} dy = \int_{-2}^{2} \frac{\left(4 - y^2\right)^2}{2} dy = \int_0^2 16 - 8y^2 + y^4 dy$$

$$= 16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \ \Big|_0^2 = \frac{256}{15}$$

Q.5) (20 Points) Find extrema and saddle points of $f(x,y)=\frac{3}{2}y^2-3xy+x^3+11$

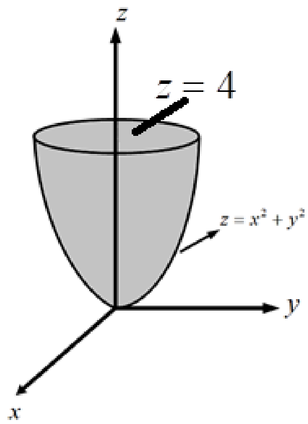
$$\left.\begin{array}{l} f_x(x,y)=-3y+3x^2=0 \\ f_y(x,y)=3y-3x=0 \end{array}\right\} \Rightarrow x=y \text{ and } y=x^2 \Rightarrow$$ critical points are $(0,0)$ *and* $(1,1)$.

$$\left.\begin{array}{l} f_{xx}(x,y)=6x \\ f_{yy}(x,y)=3 \\ f_{xy}(x,y)=0 \end{array}\right\} \Rightarrow D(x,y)=f_{xx}(x,y)f_{yy}(x,y)-\left[f_{xy}(x,y)\right]^2=18x$$

Critical points	The value of $D(x,y)$	The value of f_{xx}	Conclusion
$(0,0)$	$-9<0$	irrelevant	Saddle point
$(1,1)$	$9>0$	$6>0$	Local Minimum

Q-6) (20 Points) Use cylindrical coordinates to evaluate $\iiint\limits_Q \sqrt{x^2+y^2}\,dV$ where Q is the region bounded by the graphs of the equations $z = x^2 + y^2$ and $z = 3$ (see the figure).

Solution:



$$Q = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi\pi \\ 0 \le r \le 2 \\ r^2 \le z \le 4 \end{array} \right\} \Rightarrow$$

$$\iiint\limits_Q \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(4r^2 - r^4\right)dr\,d\theta = \int_0^{2\pi}\left[\frac{4}{3}r^3 - \frac{r^5}{5}\right]_0^{2\pi} d\theta = \int_0^{2\pi}\frac{64}{15}\,d\theta = \frac{128}{15}\pi$$