Faculty of Arts and Sciences
Department of Mathematics
MATH152 - Midterm I
16.04.2018

Std.: **Name –Surname** **Gr:**

1	2	3	4	5	6	TOTAL

Duration : 90 mins

----- GOOD LUCK-----

Q-1) (20 points) Find the center, radius and interval of convergence for

$$\sum_{n=1}^{\infty}\frac{(x+1)^n}{n5^n}$$

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}\,n5^n}{(n+1)5^{n+1}(x+1)^n}\right| = \frac{|x+1|}{5},$$

1) If $\frac{|x+1|}{5} < 1 \Rightarrow |x+1| < 5, \Rightarrow -6 < x < 4$ it is convergent.
2) If $x < -6 \ or \ x > 4$ it is divergent.
3) Assume that $x = -6 \Rightarrow \sum_{n=1}^{\infty}\frac{(x+1)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ (convergent from alternating Series test)
4) Assume that $x = 4 \Rightarrow \sum_{n=1}^{\infty}\frac{(x+1)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ (divergent from p-test)

Center of convergence : $x = -1$,
Radius of convergence : $R = 5$,
Interval of convergence : $[-6, 4)$.

Q-2) (20 points) Use power series representations to evalaute the following limit

$$\lim_{x\to 0}\frac{x\sin(2x)}{1-\cos(3x)}$$

<u>Hint</u>: (Do not use L'Hopital Rule)

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},\quad -\infty<x<\infty.$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\quad -\infty<x<\infty$$

<u>**Solution:**</u>

$$\sin 2x = 2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots+(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{(2n+1)!},\quad -\infty<x<\infty.$$

$$\cos 3x = 1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}x^{2n}}{(2n)!},\quad -\infty<x<\infty$$

$$\lim_{x\to 0}\frac{x\sin(2x)}{1-\cos(3x)}=\lim_{x\to 0}\frac{x(2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots+(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}+\cdots)}{1-(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots)}$$

$$\lim_{x\to 0}\frac{x\sin(2x)}{1-\cos(3x)}=\lim_{x\to 0}\frac{2x^2-\frac{8x^4}{3!}+\frac{32x^6}{5!}-\cdots+(-1)^n\frac{2^{2n+1}x^{2n+2}}{(2n+1)!}+\cdots}{\frac{9x^2}{2!}-\frac{3^4x^4}{4!}-\cdots+(-1)^n\frac{(3x)^{2n}}{(2n)!}+\cdots)}$$

$$\lim_{x\to 0}\frac{x\sin(2x)}{1-\cos(3x)}=\lim_{x\to 0}\frac{x^2(2-\frac{8x^2}{3!}+\frac{32x^4}{5!}-\cdots+(-1)^n\frac{2^{2n+1}x^{2n}}{(2n+1)!}+\cdots}{x^2(\frac{9}{2!}-\frac{3^4x^2}{4!}-\cdots+(-1)^n\frac{(3x)^{2n-2}}{(2n)!}+\cdots)}=\frac{4}{9}.$$

Q.3) Given that $\vec{u} = \langle 2, -2, 1 \rangle$ and $\vec{v} = \langle 1, 3, 2 \rangle$.

Solution:

a) (4 points) Find the dot product $\vec{u} \cdot \vec{v}$

$\vec{u} \cdot \vec{v} = 2 \cdot 1 - 2 \cdot 3 + 1 \cdot 2 = -2$

b) (6 points) Find the cross product $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -7i - 3j + 8k$$

c) (10 points) Find the vector projection of $2\vec{u}$ on $\vec{u} + \vec{v}$.

$2\vec{u} = \langle 4, -4, 2 \rangle$ and $\vec{u} + \vec{v} = \langle 3, 1, 3 \rangle$

$$proj_{\vec{u}+\vec{v}} 2\vec{u} = \frac{2\vec{u} \cdot (\vec{u} + \vec{v})}{|\vec{u} + \vec{v}|^2} (\vec{u} + \vec{v}) = \frac{14}{19} \langle 3, 1, 3 \rangle = \langle \frac{42}{19}, \frac{14}{19}, \frac{42}{19} \rangle$$

Q.4) a) (10 points) Find all second order partial derivatives of $f(x,y)=x\ln(xy)$.

Solution:

$$f_x(x,y)=\ln(xy)+1, \qquad f_y(x,y)=\frac{x}{y}$$

$$f_{xx}(x,y)=\frac{1}{x}, \qquad f_{xy}(x,y)=\frac{1}{y}=f_{yx}(x,y), \qquad f_{yy}(x,y)=-\frac{x}{y^2}$$

b) (10 points) Use Chain rule to evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ where $w=\sin(xy)-x\sin y$, $x=u^2+v^2$ and $y=5u^3$.

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}=2u(y\cos(xy)-\sin y)+15u^2(x\cos(xy)-x\cos y)$$

$$=2u[\ 5u^3\cos(5u^3(u^2+v^2))-\sin 5u^3\ ]+15u^2(u^2+v^2)[\ \cos 5u^3(u^2+v^2)-\cos 5u^3\ ]$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}=2v(y\cos(xy)-\sin y)=2v[\ 5u^3\cos(5u^3(u^2+v^2))-\sin 5u^3\ ]$$

Q-5) a) (10 points) Find the equation of the line through points $P(3,-2,4)$ and $Q(2,1,-3)$.

Solution: $\overrightarrow{PQ} = \langle -1, 3, -7 \rangle$

$$\text{Line} \quad L: \begin{cases} x = 3 - t, \\ y = -2 + 3t, \quad -\infty < t < \infty \\ z = 4 - 7t, \end{cases}$$

b) (10 points) Is the point $\mathrm{R}(4,-5,11)$ lies on the line through $P(3,-2,4)$ and $Q(2,1,-3)$.

$$t = -1 \Rightarrow \begin{cases} x = 3 + 1 = 4 \\ y = -2 - 3 = -5 \\ z = 4 + 7 = 11 \end{cases} \Rightarrow \text{YES } R(4,-5,11) \text{ lies on the line } L.$$

Q-6) a) (10 points) a) Find the intersection point of the line $x = 1 + 2t,\quad y = 3 - t,$ $z = 3 + 5t$ and the plane $2x + 3y + z = 2$.

Solution:

$$2(1 + 2t) + 3(3 - t) + 3 + 5t = 2$$
$$6t + 14 = 2$$
$$t = -2$$

$\begin{cases} x = 1 - 4 = -3 \\ y = 3 + 2 = 5 \\ z = 3 - 10 = -7 \end{cases}$ The intersection point is $P(-3, 5, -7)$.

b) (10 points) If the line $x = 1 + (a + 2)t,\quad y = 1 - bt,\quad z = -1 + (c - 1)t$ and the plane $-x + 2y - 3z = 0$ are orthogonal to each other, find a, b and c.

Solution:

$\vec{n} = \langle -1, 2, -3 \rangle \rightarrow a + 2 = -1, -b = 2, c - 1 = -3$

$a = -3,\quad b = -2,\quad c = -2.$