Math 152 Midterm Exam Solution

2015-2016 Spring Semester

EMU
Faculty of Arts and Sciences

Department of Mathematics

Faculty of Arts and Sciences
Department of Mathematics
MATH152
Midterm I

Std.: **Name –Surname** **Gr:** **08.04.2016**

1	2	3	4	5	6	TOTAL

Duration : 90 mins

Q-1) (20 points) Find the center, radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n n}$

$a_n = \frac{(x-4)^n}{3^n n} \quad a_{n+1} = \frac{(x-4)^{n+1}}{3^{n+1}(n+1)}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{3^{n+1}(n+1)}\frac{3^n n}{(x-4)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x-4)}{3}\right| = \left|\frac{(x-4)}{3}\right|$ for absolute convergence $\left|\frac{(x-4)}{3}\right| < 1 \Rightarrow |x-4| < 3$

$\Rightarrow -3 < x-4 < 3 \Rightarrow 1 < x < 7$

$if \;\; x = 1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-3)^n}{3^n n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ $\Rightarrow$Converges from alternating series test

$if \;\; x = 7 \Rightarrow \sum_{n=1}^{\infty}\frac{(3)^n}{3^n n} = \sum_{n=1}^{\infty}\frac{1}{n}$ $\Rightarrow$Diverges from p- series test. Thus the interval of convergence is $[1,7)$, radius of convergence is 3, center is 4.

Q-2) a) (12 points) Find the parametric equation of the line through the point $A(0,2,-3)$ that is parallel to the line through points $B(3,-1,2)$ and $C(2,1,4)$.

$\vec{u} = \overrightarrow{BC} = \langle -1,2,2 \rangle$ and $A = (0,2,-3)$

$$Eqn: \frac{x-0}{-1} = \frac{y-2}{2} = \frac{z+3}{2} = t \Rightarrow \begin{cases} x = -t \\ y = 2t+2, \quad -\infty < t < \infty \\ z = 2t-3 \end{cases}$$

b) (8points) Determine whether the line obtained in part a) and the plane $3x + y + z = 0$ intersect each other. If so find the point of intersection.

$$3(-t) + (2t+2) + (2t-3) = 0 \Rightarrow -3t + 2t + 2t = 1 \Rightarrow t = 1$$

$$\Rightarrow x = -1, \quad y = 4, \quad z = -1 \Rightarrow (-1,4,-1)$$

Q-3) a)(15 points) Find the equation of the plane through points $P(2,1,0)$, $Q(3,1,-1)$ and $R(2,-1,1)$.

$$\overrightarrow{PQ} = \langle 1,0,-1 \rangle, \quad \overrightarrow{PR} = \langle 0,-2,1 \rangle$$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & -2 & 1 \end{vmatrix} = -2\vec{i} - \vec{j} - 2\vec{k}, \quad P = (2,1,0)$$

$$\Rightarrow Eqn. \ -2(x-2) + (-1)(y-1) - 2z = 0$$

$$\Rightarrow -2x - y - 2z = -5$$

b) (5 points) Show that the point $A(1,1,1)$ is on the plane obtained in part a).

$\Rightarrow -2(1) - 1 - 2 = -5 \Rightarrow$ Therefore $A(1,1,1)$ is on the plane

Q.4) a) (14 points) Use the power series representation of

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad (-1 < x < 1)$$

to find power series representation of $f(x) = \ln(1-x)$ in powers of x.

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots x^n + \ldots$$

$$-\frac{1}{1-x} = -1 - x - x^2 - \ldots - x^n - \ldots$$

$$\int -\frac{1}{1-x}\,dx = \ln(1-x) + c = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^{n+1}}{n+1} - \ldots \text{ for } -1 < x < 1$$

$$x = 0 \Rightarrow \ln 1 + c = 0 \Rightarrow c = 0 \Rightarrow \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^{n+1}}{n+1} - \ldots = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

b) (6 points) Use the first 3 non-zero terms of the power series representation of $\ln(1-x)$ to find the approximate value of $\ln\left(\frac{1}{2}\right)$.

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

$$x = \frac{1}{2} \Rightarrow \ln\left(\frac{1}{2}\right) = -\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} = -\frac{2}{3}$$

Q-5) Given $\vec{u} = 2i - j + 5k$, $\vec{v} = i + 2j + 2k$ and $\vec{w} = -i + 3j + k$ then find

a) (10 points) $(\vec{u} + 2\vec{v}) \cdot \vec{w}$.

$$\vec{u} + 2\vec{v} = \langle 2, -1, 5 \rangle + \langle 2, 4, 4 \rangle = \langle 4, 3, 9 \rangle$$

$$(\vec{u} + 2\vec{v}).\vec{w} = \langle 4, 3, 9 \rangle . \langle -1, 3, 1 \rangle = 14$$

b) (10 points) Vector projection of $\vec{u}$ onto $\vec{v}$.

$$\text{Proj}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u}.\vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \frac{10}{9} \langle 1, 2, 2 \rangle = \left\langle \frac{10}{9}, \frac{20}{9}, \frac{20}{9} \right\rangle$$

Q-6) Let C be the curve represented by $r(t) = \cos t i + 2tj + \sin tk,\ \ 0 \le t \le 2\pi$.
Then find

a) (6 points) $r'(t)$.

$$r'(t) = -\sin t\vec{i} + 2\vec{j} + \cos t\vec{k}$$

b) (7 points) Arc length of C between $0 \le t \le 2\pi$.

$$ArcLength = \int_0^{2\pi} \sqrt{(-\sin t)^2 + 2^2 + (\cos t)^2} = \int_0^{2\pi} \sqrt{5}dt = \sqrt{5}t\Big|_0^{2\pi} = 2\sqrt{5}\pi$$

c) (7 points) Unit tangent vector to the curve C at $t = \frac{\pi}{2}$.

$$r'(t) = -\sin t\vec{i} + 2\vec{j} + \cos t\vec{k}$$

$$\|r'(t)\| = \sqrt{\sin^2 t + 4 + \cos^2 t} = \sqrt{5}$$

$$\frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{5}}\left(-\sin t\vec{i} + 2\vec{j} + \cos t\vec{k}\right)$$

$$\frac{r'\left(\frac{\pi}{2}\right)}{\left\|r'\left(\frac{\pi}{2}\right)\right\|} = \frac{1}{\sqrt{5}}\left(-\sin\frac{\pi}{2}\vec{i} + 2\vec{j} + \cos\frac{\pi}{2}\vec{k}\right) = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \right\rangle$$