Eastern Mediterranean University

Calculus II

Final Examination Solution

Department of Mathematics

2017-18 **FALL**

Faculty of Arts and Sciences
Department of Mathematics
MATH152 - Midterm I

Std.No: **Name –Surname** **16.11.2017**

1	2	3	4	5	6	TOTAL

Duration : 90 mins **Gr. No:**

Q-1) (20 points) Find the center, radius and interval of convergence for $\sum_{n=1}^{\infty}\frac{(2x-3)^n}{n}$

$$a_n=\frac{(2x-3)^n}{n},\ a_{n+1}=\frac{(2x-3)^{n+1}}{n+1}\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}n}{(n+1)(2x-3)^n}\right|=|2x-3|.$$

i) $|2x-3|<1\Rightarrow 1<x<2,\quad \textit{absolutely convergent}\Rightarrow \textit{convergent.}$

ii) $|2x-3|>1\Rightarrow x>2,\ or\ x<1,\quad \Rightarrow \textit{divergent.}$

iii) $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{(2x-3)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\Rightarrow \textit{convergent from alternating series test}$

$x=2\Rightarrow\sum_{n=1}^{\infty}\frac{(2x-3)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\Rightarrow \textit{divergent (harmonic series).}$

Center of convergence : $x=\frac{3}{2}.$

Interval of Convergence : $[1,2)$

Radius of convergence : $R=\frac{1}{2}.$

Q-2) (20 points) Find the equation of the plane through points $P(2,3,1)$, $Q(1,4,2)$ and $R(2,4,3)$.

$\overrightarrow{PQ} = \langle -1,1,1 \rangle, \quad \overrightarrow{PR} = \langle 0,1,2 \rangle.$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = i + 2j - k$$

The equation of the plane: $\quad x + 2y - z = 7$

Q-3) (20 points) Let l_1 be the line with parametric equation $x = 1 + 2t, \quad y = 4 - t, \quad z = 5 - 3t, \quad -\infty < t < \infty,$ and let l_2 be the line which passes through the points $P(3,4,-2)$ and $Q(-1,6,4)$.

a) Determine whether the point $R(5,2,-1)$ lies on the line $l_{1.}$.

b) Find the parametric equation of the line l_2.

c) Determine whether two lines, $l_{1.}$ and l_2 are parallel to each other.

a) Since $t = 2$, is a solution of $5 = 1 + 2t, \;\; 2 = 4 - t, \;\; -1 = 5 - 3t,$
Therefore, $R(5,2,-1)$ is a point on the line $l_{1.}$.

b) Direction vector, $n = \overrightarrow{PQ} = \langle -4,2,6 \rangle$. The equation of the line is
$x = 3 - 4t, \; y = 4 + 2t, \; z = -2 + 6t, \quad -\infty < t < \infty.$

c) <u>**1st way:**</u>

Direction vectors of l_1 and l_2 are $n_{l_1} = \langle 2,-1,-3 \rangle$ and $n_{l_2} = \langle -4,2,6 \rangle$ respectively.
$n_{l_2} = -2n_{l_1}$ therefore $n_{l_2} \parallel n_{l_1}$ and two lines are parallel.

<u>**2nd way**</u>: $n_{l_2} \times n_{l_1} = 0$ implies that $n_{l_2} \parallel n_{l_1}$ thus lines are parallel.

Q.4) (20 points)

a) Find the Maclaurin series of e^{x^3} (series in powers of x).

b) Use the power series representation of e^{x^3} obtained in section (a) and $P_6(x)$ to find the approxiate value of $e^{\frac{1}{2}}$.

c) Use the power series representation to evaluate the following limit.

$$\lim_{x\to 0} \frac{e^{x^3}-1}{x(\cos x-1)}$$

(Hint : $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for all x)

a) $e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!} + \cdots \quad -\infty < x < \infty.$

b) $P_6(x) = 1 + x + \frac{x^6}{2}$, the approximate value of $e^{\frac{1}{2}}$ by using $P_6(x)$ is the following.

$$e^{\frac{1}{2}} \approx P_6(\sqrt[3]{\frac{1}{2}}) = 1 + \frac{1}{2} + \frac{1}{8} = \frac{13}{8}.$$

c) $$\lim_{x\to 0} \frac{e^{x^3}-1}{x(\cos x-1)} = \lim_{x\to 0} \frac{(1 + x^3 + \frac{x^6}{2} + \cdots) - 1}{x[(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots) - 1]}$$

$$= \lim_{x\to 0} \frac{x^3 + \frac{x^6}{2} + \cdots}{x(-\frac{x^2}{2} + \frac{x^4}{24} - \cdots)} = \lim_{x\to 0} \frac{x^3 + \frac{x^6}{2} + \cdots}{-\frac{x^3}{2} + \frac{x^5}{24} - \cdots} = \frac{1}{-\frac{1}{2}} = -2$$

Q-5) (20 points) Given $\vec{u} = 2i - 1j + 3k$ and $\vec{v} = 3i - j - k$ then find

a) $\vec{u} \cdot \vec{v}$,

b) The cosine of the angle between $\vec{u}$ and $(\vec{u} + \vec{v})$

c) the unit vector in the direction of $\vec{u} + \vec{v}$.

a) $\vec{u} \cdot \vec{v} = 6 + 1 - 3 = 4$.

b) $\vec{u} + \vec{v} = \langle 5, -2, 2 \rangle \Rightarrow \|\vec{u} + \vec{v}\| = \sqrt{33}$, $\quad (\vec{u} + \vec{v}) \cdot \vec{u} = 18$

$$\cos\theta = \frac{(\vec{u} + \vec{v}) \cdot \vec{u}}{\|\vec{u} + \vec{v}\| \|\vec{u}\|} = \frac{18}{\sqrt{14}\sqrt{33}} .$$

c) The unit vector in the direction of $\vec{u} + \vec{v}$ is $\dfrac{\vec{u} + \vec{v}}{\|\vec{u} + \vec{v}\|} = \left\langle \dfrac{5}{\sqrt{33}}, \dfrac{-2}{\sqrt{33}}, \dfrac{2}{\sqrt{33}} \right\rangle$

Q-6) (20 points) a) **(10 points)** Find all second order partial derivatives of $f(x, y) = (x^2 + 3y)^3$

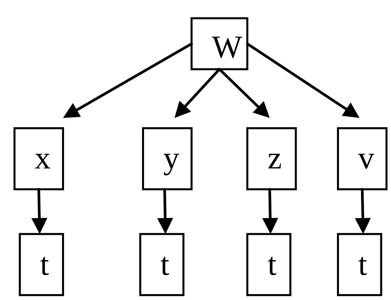
$$f_x(x, y) = 6x(x^2 + 3y)^2 , \qquad f_y(x, y) = 9(x^2 + 3y)^2$$

$$f_{xx}(x, y) = 6(x^2 + 3y)^2 + 24x^2(x^2 + 3y),$$

$$f_{xy}(x, y) = 36x(x^2 + 3y) = f_{yx}(x, y),$$

$$f_{yy}(x, y) = 54(x^2 + 3y)$$

b) **(10 points)** Let $w = (xy^2z + v)$, $x = t,\ y = t^2,\ z = e^t,\ \mathrm{v} = \ln t$. Then use chain rule to find $\dfrac{dw}{dt}$.



$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt} + \frac{dw}{dz}\frac{dz}{dt} + \frac{dw}{dv}\frac{dv}{dt}$$

$$\frac{dw}{dt} = y^2 z + 4xyxt + xy^2 e^t + \frac{1}{t}$$

$$= 5t^4 e^t + t^5 e^t + \frac{1}{t}$$