CALCULUS II

MIDTERM EXAM SOLUTION

FALL 2016-2017

EMU
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS

Faculty of Arts and Sciences
Department of Mathematics
MATH152 - Midterm I

Std.No: **Name –Surname** **18.11.2016**

1	2	3	4	5	6	TOTAL

Duration : 90 mins **Gr. No:**

Q-1) (20 points) Find the center, radius and interval of convergence for $\sum_{n=1}^{\infty}\frac{(x-5)^n}{3n^2}$

1. Center of convergence $x-5=0 \Rightarrow x=5$

2. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{3n^2}$ $a_n=\frac{(x-5)^n}{3n^2}$, $a_{n+1}=\frac{(x-5)^{n+1}}{3(n+1)^2} \Rightarrow$ by using absolute ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{3(n+1)^2}\frac{3n^2}{(x-3)^n}\right|=\lim_{n\to\infty}\left|\frac{(x-5)}{3(n+1)^2}3n^2\right|=|x-5|$$

i) If $|x-5|<1 \Rightarrow -1<x-5<1 \Rightarrow 4<x<6$ absolutely convergent $\Rightarrow$ therefore convergent

ii) $x<4$ or $x>6$ divergent

iii) $x=4 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{3n^2} \Rightarrow \left\{\begin{array}{l} a_n=\frac{1}{3n^2}>0 \\ a_{n+1}=\frac{1}{3(n+1)^2}<\frac{1}{3n^2}=a_n \\ \lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{3n^2}=0 \end{array}\right\} \Rightarrow$ converges from alternating series test

$x=6 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{3n^2}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow$ converges from p-series test,

constant multiple of a convergent series is convergent

Interval of convergence: $[4,6]$

Radius of convergence: $R=1$

Q-2) (20 points) Find the equation of the plane through points $P(1,1,0)$, $Q(2,3,-1)$ and $R(3,1,1)$.

$\overrightarrow{PQ}=\langle 1,2,-1\rangle$, $\overrightarrow{PR}=\langle 2,0,1\rangle$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & -1\\ 2 & 0 & 1\end{vmatrix}=\langle 2,-3,-4\rangle \Rightarrow \text{ The equation of the plane is}$$

$$2(x-1)-3(y-1)-4(z-0)=0$$

$$\Rightarrow 2x-3y-4z+1=0$$

Q-3) (20 points)

a) Find the parametric equation of the line through $P(0,-3,2)$ and $Q(4,5,0)$.
b) Find the intersection point of the line obtained in section a) and the plane $x+y+z=4$.

$$a)\ \overrightarrow{PQ}=\langle 4,8,-2\rangle \Rightarrow \begin{cases} x=4t \\ y=-3+8t, \quad -\infty<t<\infty \\ z=2-2t \end{cases}$$

$$b)\ 4t-3+8t+2-2t=4 \Rightarrow 10t=5 \Rightarrow t=\frac{1}{2} \Rightarrow \begin{cases} x=4\frac{1}{2}=2 \\ y=-3+8\frac{1}{2}=1 \\ z=2-2\frac{1}{2}=1 \end{cases}$$

$\Rightarrow R(2,1,1)$ is the intersection point.

Q.4) (20 points)

a) Find the Maclaurin series of $f(x)=\ln(1+x^2)$ in powers of x.

b) Find the Taylor series of $f(x)=\dfrac{1}{x}$ in powers of $x-1$.

$$a)\ \frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \left(\text{converges for } |x|<1\right)\Rightarrow \frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow \frac{2x}{1+x^2}=2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$

$$\Rightarrow \int \frac{2x}{1+x^2}dx=2\sum_{n=0}^{\infty}(-1)^n\int x^{2n+1}dx \Rightarrow \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{n+1}\Rightarrow x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow$$

$$\ln\left(1+x^2\right)+c=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n}\Rightarrow \text{take } x=0\Rightarrow \ln 1+c=0\Rightarrow c=0$$

$$\Rightarrow \ln\left(1+x^2\right)=x^2\sum_{n=0}^{\infty}(-1)^n x^{2n},\ \ -1<x<1,$$

$$b)\ \frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n\Rightarrow \frac{1}{x}=\frac{1}{1+(x-1)}=\sum_{n=0}^{\infty}(-1)^n(x-1)^n\Rightarrow\left(\text{converges for } |x-1|<1\right)$$

Q-5) (20 points) Given $\vec{u}=3i+2j+k$ and $\vec{v}=2i+j+2k$ then find

a) $\|\vec{u}\times\vec{v}\|$.

b) the projection of $\vec{u}$ onto $\vec{v}$.

c) the unit vector in the direction of $\vec{u}\times\vec{v}$.

$$a)\ \vec{u}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 2 & 1\\ 2 & 1 & 2\end{vmatrix}=3\vec{i}-4\vec{j}-\vec{k}\Rightarrow \|\vec{u}\times\vec{v}\|=\sqrt{9+16+1}=\sqrt{26}$$

$$b)\ \operatorname{Proj}_{\vec{v}}\vec{u}=\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}=\frac{6+2+2}{\left(\sqrt{2^2+1^2+2^2}\right)^2}(2,1,2)=\frac{20}{9}\vec{i}+\frac{10}{9}\vec{j}+\frac{10}{9}\vec{k}$$

$$c)\ \frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|}=\frac{3}{\sqrt{26}}\vec{i}-\frac{4}{\sqrt{26}}\vec{j}-\frac{1}{\sqrt{26}}\vec{k}$$

Q-6) (20 points) Let C be the curve represented by the vector-valued function

$r(t) = (2\cos t)\ i + (2\sin t)\, j + \sqrt{5}\, tk.$ Then find

a) $r(0)$.

b) unit tangent vector to the curve C at $t = \pi$.

c) the arc length of the portion of C where $0 \le t \le \pi$

$$a)\ r(0) = (2\cos 0)\ \vec{i} + (2\sin 0)\vec{j} + \sqrt{5}\left(0\right)\vec{k} = 2\vec{i}$$

$$b)\ r'(\mathrm{t}) = -2\sin t\ \vec{i} + 2\cos t\vec{j} + \sqrt{5}\vec{k}$$

$$\left\| r'(\mathrm{t}) \right\| = \sqrt{\left(-2\sin t\right)^2 + \left(2\cos t\right)^2 + 5} = \sqrt{9} = 3$$

$$\frac{r'(\mathrm{t})}{\left\| r'(\mathrm{t}) \right\|} = \frac{-2\sin t}{3}\vec{i} + \frac{2\cos t}{3}\vec{j} + \frac{\sqrt{5}}{3}\vec{k}$$

$$\frac{r'(\pi)}{\left\| r'(\pi) \right\|} = \frac{-2\sin \pi}{3}\vec{i} + \frac{2\cos \pi}{3}\vec{j} + \frac{\sqrt{5}}{3}\vec{k} = -\frac{2}{3}\vec{j} + \frac{\sqrt{5}}{3}\vec{k}$$

$$c)\ \text{Arc Length} = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{\left(-2\sin t\right)^2 + \left(2\cos t\right)^2 + \left(\sqrt{5}\right)^2}\, dt =$$

$$\int_0^\pi \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 5}dt = \int_0^\pi \sqrt{9}dt = 3t\Big|_0^\pi = 3\pi$$