Faculty of Arts and Sciences
Department of Mathematics
MATH152, Calculus II – Final Exam

Std. No: **Name :** **Surname :** **28.05.2018**

1	2	3	4	5	6	TOTAL

Duration : 120 mins

Q-1) Given $F(x,y,z) = x^2yz\, i + 3xz\, j + 4yz^2\, k$. Then evaluate

a) (7 points) $div\, F = \nabla \cdot F$

$$div\, F = \nabla \cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 2xyz + 0 + 8yz = 2xyz + 8yz$$

b) (8 points) $curl\, F = \nabla \times F$

$$curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 3xz & 4yz^2 \end{vmatrix} = i\,(4z^2 - 3x) - j\,(0 - x^2y) + k\,(3z - x^2z)$$

$$= (4z^2 - 3x)i + (x^2y)j + (3z - x^2z)k$$

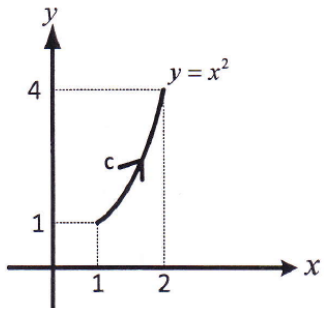
c) (5 points) Is $F(x,y,z) = x^2yz\, i + 3xz\, j + 4yz^2\, k$ conservative? Explain why.

$curl\, F = (4z^2 - 3x)i + (x^2y)j + (3z - x^2z)k \neq 0$ hence F is not conservative

Q-2) Evaluate the line integral

$$\int_C xy^2\,dx + 2x^2y\,dy$$

where C is a portion of the parabola $y = x^2$ from the point $(1,1)$ to the point $(2,4)$.



Let: $x = t$, then $y = t^2$, $dx = dt$ and $dy = 2t\,dt$
where $1 \le t \le 2$.

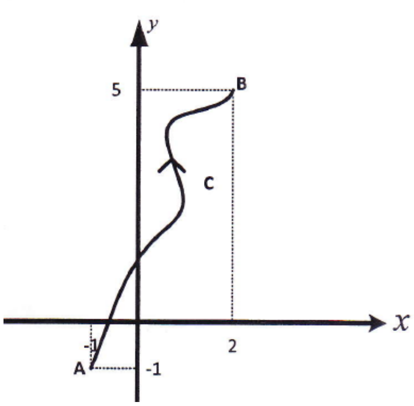
$$\int_C xy^2\,dx + 2x^2y\,dy = \int_1^2 (t)(t^2)^2\,dt + 2(t)^2(t^2)\,2t\,dt = \int_1^2 5t^5\,dt$$

$$= \left[\frac{5t^6}{6}\right]_1^2 = \frac{320}{6} - \frac{5}{6} = \frac{315}{6}$$

Q-3) (20 Points) Evaluate the line integral

$$\int_C (2+y)\,dx + (1+x)\,dy$$

where C is a curve from the point $A(-1,-1)$ to the point $B(2,5)$. (see the figure)



$$\int_C (2+y)\,dx + (1+x)\,dy = \int_C F \cdot dr$$

where $F = \underbrace{(2+y)}_{M}\, i + \underbrace{(1+x)}_{N}\, j$

$$\frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}$$

F is conservative. $F = \nabla f$, f is the potential.

$$F = \underbrace{(2+y)}_{M}\, i + \underbrace{(1+x)}_{N}\, j = \frac{\partial f}{\partial x}\, i + \frac{\partial f}{\partial y}\, j$$

$$\frac{\partial f}{\partial x} = 2 + y \;\rightarrow f = \int 2 + y\,dx = 2x + yx + C$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial C}{\partial y} = 1 + x \;\rightarrow\; \frac{\partial C}{\partial y} = 1 \;\rightarrow C = \int 1\,dy = y + K$$

$$f = 2x + yx + y + K$$

$$\int_C (2+y)\,dx + (1+x)\,dy = f(2,5) - f(-1,-1)$$

$$= [2(2) + (5)(2) + (5) + K] - [2(-1) + (-1)(-1) + (-1) + K]$$

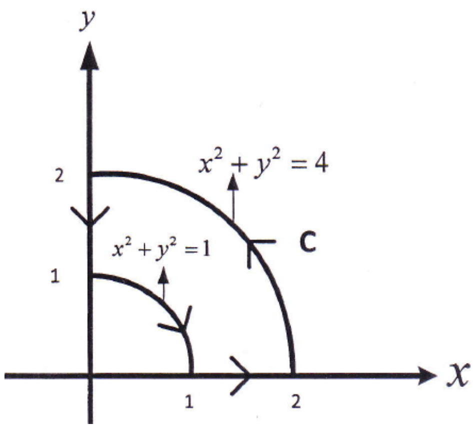
$$= [19 + K] - [-2 + K] = 21$$

An alternative method is to use the independence of path and calculate the line integral along the straight line from $(-1,-1)$ to $(2,5)$...

Q-4) (20 points) Use Green's Theorem to evaluate the line integral

$$\oint_C (-yx^2)\,dx + (xy^2 + y^3)\,dy$$

where C is the curve given in the figure.



$$\oint_C \underbrace{(-yx^2)}_{M}\,dx + \underbrace{(xy^2 + y^3)}_{N}\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

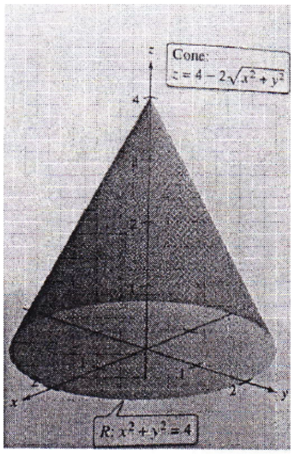
$$= \iint_R (y^2 + x^2)\,dA = \int_0^{\pi/2} \int_1^2 r^2\, r dr d\theta$$

$$= \int_0^{\pi/2} \int_1^2 r^3\, dr d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_1^2 d\theta = \int_0^{\pi/2} \frac{15}{4} d\theta = \frac{15\pi}{8}$$

Q.5) (20 Points) Evaluate the surface integral

$$\iint_S \left(\sqrt{x^2+y^2}+4-z\right) dS$$

where S is the surface given by $z = 4 - 2\sqrt{x^2+y^2}$ with $z \geq 0$. (see figure)



$$G = z - \left(4 - 2\sqrt{x^2+y^2}\right)$$

$$\nabla G = i\left(\frac{-2x}{\sqrt{x^2+y^2}}\right) + j\left(\frac{-2y}{\sqrt{x^2+y^2}}\right) + k$$

$$|\nabla G| = \sqrt{\frac{4(x^2+y^2)}{x^2+y^2}+1} = \sqrt{5}$$

$$\iint_S \left(\sqrt{x^2+y^2}+4-z\right) dS = \iint_R \left(\sqrt{x^2+y^2}+4-\left(4-2\sqrt{x^2+y^2}\right)\right) |\nabla G|\, dA$$

$$= \iint_R \left(3\sqrt{x^2+y^2}\right)\sqrt{5}\ dA$$

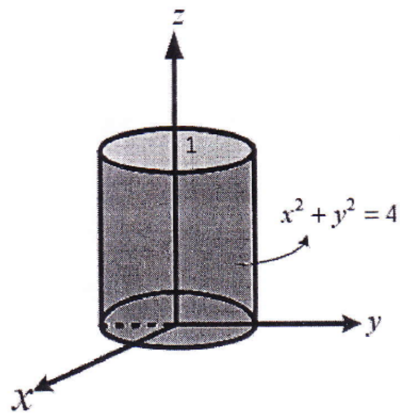
$$= \sqrt{5}\int_0^{2\pi}\int_0^{2} 3r\ r dr d\theta = \sqrt{5}\int_0^{2\pi}\int_0^{2} 3r^2\, dr d\theta$$

$$= \sqrt{5}\int_0^{2\pi} [r^3]_0^2\ d\theta = \sqrt{5}\int_0^{2\pi} 8\ d\theta = 16\sqrt{5}\,\pi$$

Q-6) (20 Points) Use Divergence Theorem to find the outward flux

$$\oiint_S F \cdot \vec{n}\ dS$$

of the vector field $F(x, y, z) = x^2\, i + y^2\, j + z^2\, k$ where S is the boundary of the closed region bounded by $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 1$. (see the figure).



$$\oiint_S F \cdot \vec{n}\ dS = \iiint_D \boldsymbol{div\ F}\ dV$$

$$\boldsymbol{div\ F} = \nabla \cdot F = 2x + 2y + 2z = 2(x + y + z)$$

$$\oiint_S F \cdot \vec{n}\ dS = 2\iiint_D (x + y + z)\, dV = 2\int_0^{2\pi}\int_0^{2}\int_0^{1} (r\cos\theta + r\sin\theta + z\,)\, rdzdrd\theta$$

$$= 2\int_0^{2\pi}\int_0^{2}\int_0^{1} (r^2\cos\theta + r^2\sin\theta + rz\,)\, dzdrd\theta$$

$$= 2\int_0^{2\pi}\int_0^{2} \left(r^2\cos\theta + r^2\sin\theta + \frac{r}{2}\right) drd\theta = 2\int_0^{2\pi} \left(\frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta + 1\right) d\theta$$

$$= 2\left[\frac{8}{3}\sin\theta - \frac{8}{3}\cos\theta + \theta\right]_0^{2\pi} = 2\left(-\frac{8}{3} + 2\pi\right) - 2\left(-\frac{8}{3}\right) = 4\pi$$