Faculty of Arts and Sciences
Department of Mathematics
MATH152, Calculus II - Final Exam **07.12.2019**

Std.No	Name –Surname	Group	Signature

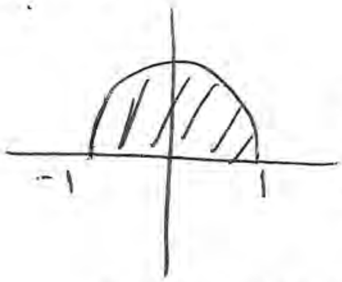
1	2	3	4	5	6	TOTAL

Duration : 120 mins

Q-1) Given $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dydx$.

a) (10 points) **Rewrite the integral** in polar coordinates.

$-1 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$



$0 \le \theta \le \pi$
$0 \le r \le 1$

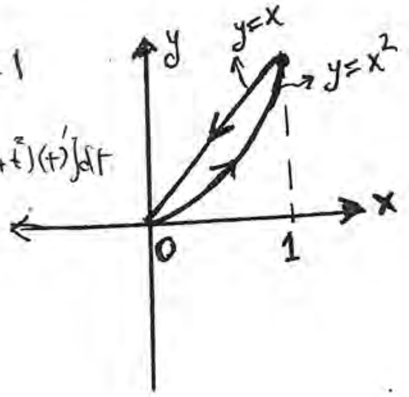
$$\int_0^\pi\int_0^1 r\, r\,dr\,d\theta = \int_0^\pi\int_0^1 r^2\,dr\,d\theta$$

b) (10 points) Evaluate the integral obtained in part a).

$$\int_0^\pi\int_0^1 r^2\,dr\,d\theta = \int_0^\pi \frac{r^3}{3}\Big|_0^1 d\theta = \int_0^\pi \frac{1}{3}\,d\theta = \frac{1}{3}\theta\Big|_0^\pi = \pi/3$$

Q-2) Given the line integral $\int_C (x+y)dx+(2y+x^2)dy$ where C is the positively oriented curve consisting of $y=x^2$ and $y=x$, $0 \le x \le 1$. (see the figure below)

a) (10 points) Evaluate the line integral by using a parametric equation of C (as a line integral).



$C = C_1 + C_2$, $C_1: \begin{cases} x=t \\ y=t^2 \end{cases}, 0 \le t \le 1$, $-C_2: \begin{cases} x=t \\ y=t \end{cases}, 0 \le t \le 1$

$\int_C = \int_{C_1} - \int_{-C_2} = \int_0^1 \left[(t+t^2)(t)' + (2t^2+t^2)(t^2)'\right]dt - \int_0^1 \left[(t+t)(t)' + (2t+t^2)(t)'\right]dt$

$= \int_0^1 (t+t^2+6t^3)dt - \int_0^1 (4t+t^2)dt =$

$= \left(\frac{t^2}{2} + \frac{t^3}{3} + \frac{6t^4}{4} - 2t^2 - \frac{t^3}{3}\right)\Big|_0^1 =$

$= \frac{1}{2} + \cancel{\frac{1}{3}} + \frac{3}{2} - 2 - \cancel{\frac{1}{3}} = 2 - 2 = \boxed{0}$

b) (10 points) Evaluate the **same** integral by using **Green's Theorem.**

$\int_C (x+y)dx + (2y+x^2)dy = \iint_G (2x-1)\,dA = \int_0^1 \int_{x^2}^{x} (2x-1)\,dy\,dx$

$= \int_0^1 (2v-1)(x-x^2)\,dx = \int_0^1 (2x^2 - 2x^3 - x + x^2)dx$

$= \int_0^1 (3x^2 - 2x^3 - x)dx = \left(x^3 - \frac{2x^4}{4} - \frac{x^2}{2}\right)\Big|_0^1 = 1 - \frac{1}{2} - \frac{1}{2} = \boxed{0}$

Q-3) Given $F(x,y)=(2xy+y^3+8x)\,i+(x^2+3y^2x+6y)\,j$.

a) (5 points) Show that $\int_C (2xy+y^3+8x)\,dx+(x^2+3y^2x+6y)\,dy$ is independent of path (F is conservative).

$$\left.\begin{array}{l} P = 2xy+y^3+8x \\ Q = x^2+3y^2x+6x \end{array}\right\} \Rightarrow P'_y = 2x+3y^2 = Q'_x \Rightarrow \bar{F} \text{ is conservative}$$

b) (10 points) Find a potential function for $F(x,y)$.

$$f'_x = 2xy+y^3+8x \Rightarrow f(x,y) = x^2y+xy^3+4x^2+c(y) \Rightarrow$$

$$\Rightarrow f'_y = x^2+3xy^2+c'(y) = x^2+3y^2x+6y \Rightarrow c'(y) = 6y \Rightarrow$$

$$\Rightarrow c(y) = 3y^2+k \Rightarrow \boxed{f(x,y) = x^2y+xy^3+4x^2+3y^2+k}$$

c) (5 points) Evaluate $\int_{(1,1)}^{(2,0)} (2xy+y^3+8x)\,dx+(x^2+3y^2x+6y)\,dy.$ =

$$= f(2,0)-f(1,1) = 4\cdot 2^2-1-1-4-3 = \boxed{7}$$

Q-4)

a) (10 points) Find $Curl\, F = \nabla \times F$, where $F(x,y,z) = (xz + y^2)\, i + (x + zy)\, j + z^2 x k$.

$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz+y^2 & x+zy & z^2x \end{vmatrix} = \left(\frac{\partial z^2 x}{\partial y} - \frac{\partial (x+zy)}{\partial z} \right) i$$

1

$$- \left(\frac{\partial z^2 x}{\partial x} - \frac{\partial (xz+y^2)}{\partial z} \right) j$$

$$+ \left(\frac{\partial (x+zy)}{\partial x} - \frac{\partial (xz+y^2)}{\partial y} \right) k$$

$$= \underbrace{(0-y)\, i}_{3} - \underbrace{(z^2-x)\, j}_{3} + \underbrace{(1-2y)\, k}_{3}$$

b) (10 points) Evaluate $\int_0^2 \int_x^{x^2} \int_0^{\sqrt{y}} z \, dz \, dy \, dx$

$$= \int_0^2 \int_x^{x^2} \left[\frac{z^2}{2} \right]_0^{\sqrt{y}} dy\, dx \quad 2$$

$$= \int_0^2 \int_x^{x^2} \frac{y}{2}\, dy\, dx \quad 1$$

$$= \int_0^2 \left[\frac{y^2}{4_{1}} \right]_x^{x^2} dx$$

$$= \int_0^2 \left(\frac{x^4}{4_{1}} - \frac{x^2}{4_{1}} \right) dx$$

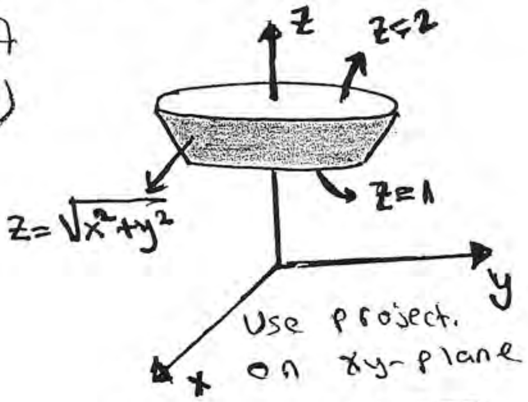
$$= \left[{}_{1}\frac{x^5}{4.5} - {}_{1}\frac{x^3}{4.3} \right]_0^2$$

$$= \frac{32}{{}_{1}20} - \frac{8}{{}_{1}12} = \frac{14}{15}$$

Q-5) (20 points) Evaluate the surface integral $\iint_S (y+1)\,dS$ where S is the surface, cut from the cone $z=\sqrt{x^2+y^2}$, by the planes $z=1$ and $z=2$. (See the figure)

$$\iint_S (y+1)\,ds = \iint (y+1)\underbrace{\sqrt{(g_x)^2+(g_y)^2+1}}_{\to\ 2}\,dA$$

$$= \iint_S (y+1)\underbrace{\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}}_{2}\,dA$$



Use project. on xy-plane

$$Z=g(x,y)=\sqrt{x^2+y^2}$$

$$\cdot\ R_{xy}=\begin{cases}0\le\theta\le 2\pi\\ 1\le r\le 2\end{cases}$$

$$y=r\sin\theta$$

1 1

$$= \int_0^{2\pi}\int_1^{2} (r\sin\theta+1)\,\sqrt{2}\cdot r\,dr\,d\theta$$

1 1 1 1 1

$$= \sqrt{2}\int_0^{2\pi}\int_1^{2}(r^2\sin\theta + r)\,dr\,d\theta \quad 1$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{r^3}{3}\sin\theta+\frac{r^2}{2}\right]_1^2 d\theta \quad 2$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{8}{3}\sin\theta+\frac{4}{2}-\left(\frac{\sin\theta}{3}+\frac{1}{2}\right)\right]d\theta \quad 2$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{7}{3}\sin\theta+\frac{3}{2}\right]d\theta$$

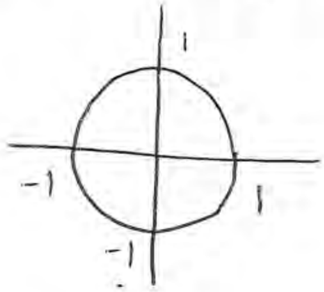
$$= \sqrt{2}\left(\frac{-7}{3}\cos\theta+\frac{3\theta}{2}\right]_0^{2\pi} \quad 2$$

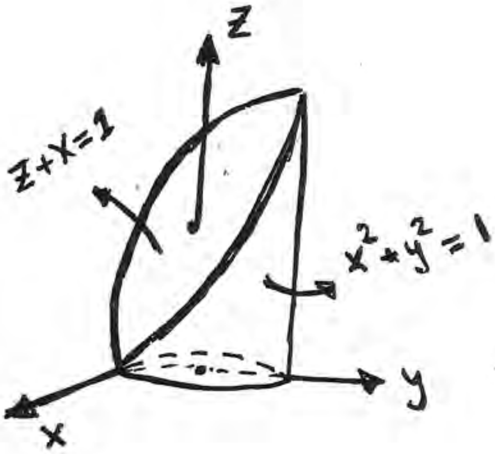
$$= \sqrt{2}\left(\frac{-7}{3}\cos 2\pi+\frac{3\cdot\cancel{2}\pi}{\cancel{2}}-\left(\frac{-7}{3}\cos 0+0\right)\right) \quad 2$$

$$= \sqrt{2}\,(3\pi) = 3\sqrt{2}\,\pi$$

Q-6) (20 points) Use Divergence Theorem to find the outward flux $\iint_S F\cdot\vec{n}\,dS$ of the vector field $F(x,y,z)=xy^2 i+yx^2 j+zk,$ where S is the boundary of the closed region bounded by $x^2+y^2=1$ and planes $z=0$ and $z+x=1.$ (see the figure given below).

Sol The projection on xy-plane is





$0 \le \theta \le 2\pi$ (5)

$0 \le r \le 1$

$0 \le z \le 1-x = 1-r\cos\theta$

By using Divergence Theorem $\iint_{S'} \vec{F}\cdot\vec{n}\,dS = \iiint_{\Omega} \nabla\cdot F\,dV$ (3)

$\nabla\cdot F = y^2+x^2+1$ (3)

$\iint_{S'} \vec{F}\cdot\vec{n}\,dS = \iiint_{\Omega} \nabla\cdot F\,dV = \iiint_{\Omega} (x^2+y^2+1)\,dV$ (2) $= \int_0^{2\pi}\int_0^{1}\int_0^{1-r\cos\theta} r(r^2+1)\,dz\,dr\,d\theta$ (3)

$= \int_0^{2\pi}\int_0^{1}\int_0^{1-r\cos\theta} (r^3+r)\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} (1-r\cos\theta)(r^3+r)\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} r^3+r-r^4\cos\theta-r^2\cos\theta\,dr\,d\theta$

$= \int_0^{2\pi} \frac{r^4}{4}+\frac{r^2}{2}-\frac{r^5}{5}\cos\theta-\frac{r^3}{3}\cos\theta\Big|_0^1\,d\theta = \int_0^{2\pi} \frac{1}{4}+\frac{1}{2}-\frac{1}{5}\cos\theta-\frac{1}{3}\cos\theta\,d\theta$

$= \int_0^{\pi} \frac{3}{4}-\frac{8}{15}\cos\theta\,d\theta = \frac{3}{4}\theta - \frac{8}{15}\sin\theta\Big|_0^{2\pi} = \left(\frac{6\pi}{4}-\frac{8}{15}\sin 2\pi\right)-(0)$

(5) $= \frac{6\pi}{4} = 3\pi/2$