Chapter 2  PROBABILITY

Key words: Sample space, sample point, tree diagram, events, complement, union and intersection of an event, mutually exclusive events; Counting techniques: multiplication rule, permutation, combination, probability of an event; Additive rules; Conditional probability, independent events, multiplicative rules, Bayes’ rule.

SAMPLE SPACE

Definition 1. Sample Space The set of all possible outcomes of a statistical experiment is called a Sample space. It is represented by the symbol S.

Definition 2. Sample Point Each outcome in a sample space is called an element or a sample point of the sample space.

Example 1. (a) Consider the experiment of tossing a coin. The sample space S of possible outcomes may be written as

\[ S = \{H, T\} \]

b) Consider the experiment of flipping a die. Then the elements of the sample space S is listed as

\[ S = \{1, 2, 3, 4, 5, 6\} \]

c) Now consider the experiment of tossing a die and then a coin once. The resultant sample space can be obtained using TREE DIAGRAM.

Therefore the sample space S is

\[ S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\} \]

d) For the experiment of flipping two coins, the sample space is \{HH, HT, TH, TT\}. This sample space has four elements.

e) For the experiment of flipping three coins, the sample space is \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}. This sample space has eight elements.

f) For the experiment of flipping \( n \) coins, where \( n \) is a positive whole number, the sample space consists of \( 2^n \) elements. There are a total of \( C(n, k) \) ways to obtain \( k \) heads and
ii) For the experiment of rolling two six-sided dice, the sample space consists of the set of the 36 possible pairings of the numbers 1, 2, 3, 4, 5 and 6.

iii) For the experiment of rolling three six-sided dice, the sample space consists of the set of the 216 possible triples of the numbers 1, 2, 3, 4, 5 and 6.

iv) For the experiment of rolling \( n \) six-sided dice, where \( n \) is a positive whole number, the sample space consists of \( 6^n \) elements.

**Definition 3. Event** An event is a subset of a sample space.

For example, \( A = \{4H, 6T\} \) is an event defined on \( S \). One can define \( 2^6 \) events on \( S \).

Empty set \( \emptyset \), is an impossible event and \( S \) is a sure event. Any subset of \( S \) is represented by capital letters such as \( A, B, C... \)

**Definition 4.**

*The Complement of an Event* The complement of an event \( A \) with respect to \( S \) is the subset of all elements of \( S \) that are not in \( A \). The complement of \( A \) is denoted by the symbol \( A' \) or \( A^c \).

*The Intersection of Events* The intersection of two events \( A \) and \( B \), denoted by the symbol \( A \cap B \) is the event containing all elements that are common to \( A \) and \( B \).

*Mutually Exclusive Events* Two events \( A \) and \( B \) are mutually exclusive or disjoint if \( A \cap B = \emptyset \), that is if \( A \) and \( B \) have no common elements.

*The Union of Events* The union of two events \( A \) and \( B \), denoted by the symbol \( A \cup B \) is the event containing all elements that belong to \( A \) or \( B \) or both.

**Important Notes.** The following results may easily be verified by means of Venn diagrams.

1. \( A \cap \emptyset = \emptyset \)
2. \( A \cup \emptyset = A \)
3. \( A \cap A' = \emptyset \)
4. \( A \cup A' = S \)
5. \( S' = \emptyset \)
6. \( \emptyset' = S \)
7. \( (A')' = A \)
8. \( (A \cap B)' = (A' \cup B') \) 1st De Morgan Rule
9. \( (A \cup B)' = A' \cap B' \) 2nd De Morgan Rule

**Example 2.** If \( S = \{0,1,2,3,4,5,6,7,8,9\} \) and \( A = \{0,2,4,6,8\} \), \( B = \{1,3,5,7,9\} \), \( C = \{2,3,4,5\} \) and \( D = \{1,6,7\} \), list the elements of the sets corresponding to the following events:

a) \( A \cup B = S \)
(b) \( A \cap B = \emptyset \) (\( A \) and \( B \) are mutually exclusive events)

(c) \( C' = \{0, 1, 6, 7, 8, 9\} \)

(d) 
\[
(C' \cap D) \cap B = \left(\{0, 1, 6, 7, 8, 9\} \cap \{1, 6, 7\}\right) \cap \{1, 3, 5, 7, 9\}
= \{1, 6, 7\} \cap \{1, 3, 5, 7, 9\}
= \{1, 7\}
\]

(e) \((S \cap C)' = C' = \{0, 1, 6, 7, 8, 9\}\)

(f) 
\[
A \cap C \cap D' = \{2, 4\} \cap \{0, 2, 3, 4, 5, 8, 9\}
= \{2, 4\}
= A \cap C
\]
COUNTING SAMPLE POINTS

**Multiplication Rule.** If an operation can be performed in \( n_1 \) ways, and if for each of these a second operation can be performed in \( n_2 \) ways, then the two operations can be performed together in \( n_1n_2 \) ways.

**Example 1.** How many breakfasts consisting of a drink and a sandwich are possible if we can select from 3 drinks and 4 kinds of sandwiches?

*Solution:* \( n_1 = 3 \) and \( n_2 = 4 \), therefore there are \( n_1n_2 = 3 \times 4 = 12 \) different ways to choose a breakfast.

**Example 2.** A certain shoe comes in 5 different styles with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs would the store have on display?

*Solution:* \( 5 \times 4 = 20 \) different pairs are available.

**Example 3.** In how many ways can a true-false test consisting of 9 questions be answered?

*Solution:* There are \( 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9 \) different answers for this test.

**Example 4.** A simple survey consists of three multiple choice questions. The first question has 3 possible answers, the second has 4 possible answers and the third has 3 possible answers. What is the total number of different ways in which this survey could be completed?

*Solution:* There are \( 3 \times 4 \times 3 = 36 \) different ways for completing the survey.

**Example 5.** How many even three-digit numbers can be formed from the digits 1,2,3,4, and 5 if
(a) each digit can be used only once?
(b) each digit can be repeated?

*Solution:* (a) Since the number must be even we have only 2 choices for the units position. For each of these there are 4 choices for hundreds position and 3 choices for tens positions. Therefore we can form a total of \( 2 \times 4 \times 3 = 24 \) even three-digit numbers.

(b) Similarly for the units position there are 2 choices, since the numbers can be repeated we have 5 choices for the hundreds and 5 choices for the tens positions. Hence, totally there are \( 2 \times 5 \times 5 = 50 \) even three-digit numbers.
Exercises

1. In how many different ways can a quiz be answered under each of the following conditions?
   a) The quiz consists of three multiple questions with four choices for each.
   b) The quiz consists of three multiple questions with four choices for each and five true-false questions.

2. A lecture hall has five doors. In how many ways a student enter the hall by one door and exit by any door?
   a) exit by a different door?
   b) exit by any door?

3. How many different seven digit telephone numbers can be formed using the digits 0,1,2,3,4,5,6,7,8,9 if the first digit cannot be zero?

Permutation. A permutation is an arrangement of all or part of a set of objects.

Theorem 1. a) The number of permutations of \( n \) distinct objects is \( n! \).
   b) The number of permutations of \( n \) distinct objects taken \( r \) at a time is
   \[
   _rP_n = \frac{n!}{(n-r)!}
   \]

Example 5. In how many ways can a Society schedule 3 speakers for 3 different meetings if they are available on any of 5 possible dates?

Solution: The total number of possible schedules is
   \[
   _3P_5 = \frac{5!}{(5-3)!} = 5.4.3 \cdot 60 = 5.4.3 \cdot 60.
   \]

Theorem 2. The number of permutations of \( n \) distinct objects arranged in a circle is \( (n-1)! \).

Theorem 3. The number of distinct permutations of \( n \) things of which \( n_1 \) are of one kind, \( n_2 \) of a second kind, …, \( n_k \) of a \( k^{th} \) kind is
   \[
   \frac{n!}{n_1!n_2!...n_k!}
   \]

Example 6. In how many different ways can 3 red, 4 yellow, and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets?

Solution: The total number of distinct arrangements is
   \[
   \frac{9!}{3!4!2!} = 1260.
   \]

Example 7. Suppose that 10 employees are to be divided among three jobs with 3 employees going to job I, 4 to job II, and 3 to job III. In how many ways can the job assignment be made?

Solution. \[
\frac{10!}{3!4!3!} = 4200
\]
Example 8. How many different ways can we rearrange the letters of MISSISSIPPI?

Solution. We have 11 letters in total, of which 4 are ‘I’, 4 are ‘S’ and ‘2’ are ‘P’. In this situation, the total number of different rearrangements is $11! / 4!4!2!$.

Theorem 4. The number of combinations of $n$ distinct objects taken $r$ at a time is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Example 8. From 4 mathematicians and 6 computer scientists, find the number of committees that can be formed consisting of 2 mathematicians and 4 computer scientists.

Solution. $\binom{4}{2} \cdot \binom{6}{4} = \frac{4!}{2!2!} \cdot \frac{6!}{4!2!} = \frac{4.3.2!}{2!} \cdot \frac{6.5.4!}{2!2.2!} = 3.6.5 = 90$

Exercises

1. In how many different ways can a union local with a membership of 25 choose a vice president and a president?

2. In a geology class each of the 4 students must write a report on one of the 8 field trips. In how many different ways can they each choose one of the field trips if
   a) no two students may choose the same field trip?
   b) there is no restriction on their choice?

3. How many four-digit serial numbers can be formed if no digit is to be repeated within any one number?

4. Seven applicants have applied for two jobs. How many ways can the jobs be filled if
   a) the first person chosen receives a higher salary than the second?
   b) there are no differences between the jobs?

5. From a group of 4 men and 5 women, how many committees of size 3 are possible
   a) with no restrictions?
   b) with 1 man and 2 women?
   c) with 2 men and 1 woman if a certain man must be on the committee?

6. In how many ways can the letters of the word ‘KRAKATOA’ be arranged?

7. How many different 3 letter words can be made using letters from the word TABLE if repetition is not allowed?

8. How many different 5 letter words can be made using letters from the word ‘MATHEMATICS’ if repetition is not allowed?
9. How many different words of 5 letters or less can be made using all the of the English alphabet if repetition is not allowed?
Probability of an Event

**Definition 1.** Suppose that an experiment has associated with it a sample space $S$. A **probability** $P$ is a numerically valued function that assigns a number $P(A)$ to every event $A$ so that the following axioms hold:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. If $A_1, A_2, \ldots$, is a sequence of mutually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for any $i \neq j$), then
   \[ P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i) \]

**Example 1.** A coin is tossed twice. What is the probability that at least one head occurs?

The sample space for this experiment is $S = \{HH, HT, TH, TT\}$. If the coin is balanced, each of these outcomes would be equally likely to occur. Therefore we assign a probability $w$ to each sample point. Then $4w = 1$ or $w = 1/4$. If $A$ represents the event of at least one head occurring, then

\[ A = \{HH, HT, TH\} \quad \text{and} \quad P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}. \]

**Theorem 1.** If an experiment can result in any one of $N$ different equally likely outcomes, and if exactly $n$ of these outcomes correspond to event $A$, then the probability of event $A$ is

\[ P(A) = \frac{n}{N} \]

**Additive Rules**

**Theorem 2.**

a) If $A$ and $B$ are two events, then

\[ P(A \cap B) = P(A) + P(B) - P(A \cup B) \]

b) If $A$ and $B$ are mutually exclusive events, then

\[ P(A \cap B) = P(A) + P(B) \]

c) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events, then

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \]

d) For three events $A$, $B$ and $C$,

\[ P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \]
Note \( P(A \setminus B) = P(A \cap B^c) = P(A) - P(A \cap B) \)

2. \( P(B \setminus A) = P(B \cap A^c) = P(B) - P(A \cap B) \)

3. \( P(A^c \cap B^c) = P(A \cup B)^c \)

4. \( P(A^c \cup B^c) = P(A \cap B)^c \).

Example 1. What is the probability of getting a total 7 or 11 when a pair of dice are tossed?

Solution. Let \( A \): event that 7 occurs; \( B \): event that 11 occurs, therefore
\[
P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}
\]

Theorem 3. If \( A \) and \( A^c \) are complementary events, then
\[
P(A) + P(A^c) = 1
\]

Example 2. In a random experiment it is known that \( P(A) = 0.35, P(A \cap B) = 0.19 \), and \( P(A^c \cap B) = 0.15 \). Calculate \( P(B) \).
Exercises

1. A pair of dice is tossed. Find the probability of getting
   (a) a total of 8
   (b) at most a total of 5.

2. Two cards are drawn in succession from a deck without replacement. What is the probability that both cards are greater than 3 and less than 9?

3. If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that
   (a) the dictionary is selected?
   (b) 2 novels and 1 book of poems are selected?

4. If 2 of the 10 employees are female and 8 male, what is the probability that exactly one female gets selected among the three?

5. A package of 6 light bulbs contain 2 defective bulbs. If 3 bulbs are selected for use, find the probability that none is defective.

6. The probability that an integrated circuit chip will have defective etching is 0.12, the probability that it will have a crack is 0.29, the probability that it will has both defects is 0.07.
   a) What is the probability that a newly manufactured chip will have either an etching or a crack defects?
   b) What is the probability that a newly manufactured chip will have neither defect?

7. Among the 24 invoices prepared by a billing department, 4 contain errors while the others do not. If we randomly check two of the invoices, what are the probabilities that
   a) both will contain errors?
   b) neither will contain an error?

8. A drum contains 3 black balls, 5 red balls and 6 green balls. If 4 balls are selected at random what is the probability that the 4 selected contain
   a) No red ball?
   b) Exactly 1 black ball?
   c) Exactly 1 red ball and exactly 2 green balls?
**Conditional Probability and Independence**

The probability of an event $B$ occurring when it is known that some event $A$ has occurred is called a *conditional probability* and is denoted by $P(B|A)$.

**Definition 1.** The conditional probability of $B$, given $A$, denoted by $P(B|A)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(A)}$$

provided that $P(A) > 0$.

**Example 1.** The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

(a) arrives on time given that it departed on time.

(b) departed on time given that it has arrived on time.

Solution. (a) $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83}$

(b) $P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{0.78}{0.82}$

**Definition 2.** Two events $A$ and $B$ are said to be *independent* if

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B).$$

This is equivalent to stating that

$$P(A \cap B) = P(A)P(B).$$

**Exercises**

1. Consider two mutually exclusive (disjoint) events, $A$ and $B$, such that $P(A) > 0$ and $P(B) > 0$. Are $A$ and $B$ independent? Give a proof for your answer.

2. Let $A$ and $B$ be two events with $P(A) = \frac{1}{3}$, $P(B') = \frac{3}{4}$ and $P(A \cup B) = \frac{1}{2}$. Find
   a) $P(A \cap B)$
   b) $P(A' \cap B')$
   c) $P(A|B')$
   d) $P(B|A)$
   e) Determine whether $A$ and $B$ are independent or not.

3. If events $A$ and $B$ are independent and $P(A) = 0.25$ and $P(B) = 0.40$, find
a) $P(A \cap B)$  
b) $P(A|B)$  
c) $P(A \cup B)$  
d) $P(A^c \cap B^c)$


**Multiplicative Rules**

**Theorem 1.** If in an experiment the events $A$ and $B$ can both occur, then

$$P(A \cap B) = P(A)P(B|A).$$

**Example 1.** One bag contains 4 white balls and 3 black balls, and second bag contains 3 white and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

**Solution.** Let $B_1, B_2$ and $W_1$ represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2 and a white ball from bag 1.

$$P\left(\left(B_1 \cap B_2\right) \cup \left(W_1 \cap B_2\right)\right) = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$

$$= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1)$$

$$= \frac{3}{7} \cdot \frac{6}{9} + \frac{4}{7} \cdot \frac{5}{9} = \frac{38}{63}.$$

**Example 2.** Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where $A_1$ is the event that the first card is a red ace, $A_2$ is the event that the second card is a 10 or a jack, and $A_3$ is the event that the third card is greater than 3 but less than 7.

**Solution.** First we define the events as

- $A_1$: the first card is a red ace
- $A_2$: the second card is a 10 or jack
- $A_3$: the third card is greater than 3 but less than 7

Then, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) = \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50} = \frac{8}{5525}.$

**Example 3.** A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

**Solution.** The sample space for the experiment consists of the 8 elements,

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Let $A$ be the event of getting 2 tails and 1 head in the 3 tosses of an unfair coin. Then

$$A = \{TTH, THH, HTT\}.$$

Assigning probabilities of $w$ and $2w$ for getting a tail and a head, respectively, we have

$$3w = 1 \text{ or } w = \frac{1}{3}. \text{ Hence } P(H) = \frac{2}{3} \text{ and } P(T) = \frac{1}{3}. \text{ Since the outcomes on each of the 3 tosses are independent, it follows that}$$
\[ = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}. \]

**Exercises**

1. In an experiment to study the relationship of hypertension and smoking habits, the following data are collected for 180 individuals:

<table>
<thead>
<tr>
<th></th>
<th>Non-smokers</th>
<th>Moderate Smokers</th>
<th>Heavy Smokers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypertension</td>
<td>21</td>
<td>36</td>
<td>30</td>
</tr>
<tr>
<td>No Hypertension</td>
<td>48</td>
<td>26</td>
<td>19</td>
</tr>
</tbody>
</table>

If one of these individuals is selected at random, find the probability that the person is
(a) experiencing hypertension, given that the person is a heavy smoker;
(b) a non-smoker, given that the person is experiencing no hypertension.

2. The probability that a married man watches a certain television show is 0.4 and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find the probability that
(a) a married couple watches the show;
(b) a wife watches the show given that her husband does;
(c) at least one person of a married couple will watch the show.

3. The probability that Ayse will be alive in 20 years is 0.7 and the probability that Ali will be alive in 20 years is 0.9. If we assume independence for both, what is the probability that neither will be alive in 20 years?

4. Assume that the events \( A \) and \( B \) satisfy
\[ P(A) = P(B) = P(B \mid A') = 1/3 \]
where \( A' \) is the complement of \( A \). What is \( P(A' \cup B) \)?
Bayes’ Rule

The following theorem, sometimes is called the **Theorem of Total Probability** or the **Rule of Elimination**.

**Theorem 1.** If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space $S$ such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event $A$ of $S$,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

**Theorem 2. Bayes’ Rule** If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space $S$, where $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event $A$ in $S$ such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

for $r = 1, 2, \ldots, k$.

**Example 1.** A group consists of 60% of girl students and 40% of boy students. 20% of boy students and 50% of girl students are Cypriots. If a student is taken at random from this group, 
(a) what is the probability that this student is Cypriot?  
(b) If at random selected student from this group is Cypriot, what is the probability that this student is a girl?

**Solution.** Consider the following events:  
G: student is a girl  
B: student is a boy  
C: student is Cypriot

$$P(G) = 0.60, \quad P(B) = 0.40, \quad P(C|B) = 0.20, \quad P(C|G) = 0.50.$$  
(a) $$P(C) = P(B)P(C|B) + P(G)P(C|G) = (0.40)(0.20) + (0.60)(0.50) = 0.38$$  
(b) $$P(G|C) = \frac{P(G \cap C)}{P(C)} = \frac{P(G)P(C|G)}{P(C)} = \frac{(0.60)(0.50)}{0.38} = 0.79.$$  

**Exercises**

1. In a certain assembly plant, three machines, $M_1, M_2$ and $M_3$ make 30%, 45% and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the product made by each machine, respectively, are defective. Find the probability that a randomly selected product is defective.
2. A lot of items contains 90% of Non-defective items. A test is used in order to detect defective items in the lot. The test gives a mistaken result (wrong result) with probability 0.05 when the item is defective and with probability 0.10 when the item is Non-defective. An item is chosen from the lot and tested. If the test says that it is Non-defective, what is the probability that the item was really Non-defective?

3. At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman?

4. Bowl A contains 2 red chips; bowl B contains two white chips; and bowl C contains 1 red chip and 1 white chip. A bowl is selected at random, and one chip is taken at random from that bowl.
   a) What is the probability of selecting a white chip?
   b) If the selected chip is not white, what is the probability that it is drawn from Bowl B?

5. A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A manufactures 25%, machine B manufactures 35% and machine C for the rest. It is known from past experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from
   (a) machine A
   (b) machine B
   (c) machine C?